Minimal surfaces:
From Soap films to Black Holes

V Jornadas Doctorales

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Principle of least action

"Nature is thrifty in all its actions"

Pierre Louis Maupertius
(Leibniz, Euler)

"You often see this phenomenon (least action) in nature because, of all possible configurations you can have, the ones that actually occur have the least energy"

L. Simon, Stanford University

Principle of least action

Minimization (or least action) has been a foundational concept in both geometry and physics.

Simple example:

The trajectory of a particle under a gravitational field (geodesic)

Principle of least action

"Geometry, the science of space, needs differential equations to measure the curvature of objects and how it changes. That is one reason curvature is so linked to physics. That is also why geometry is instrumental to so many areas of physics"

S.T. Yau, Harvard University

Principle of least action

In the 18th century, Euler, Lagrange, Meusnieur studied a natural problem on minimization:

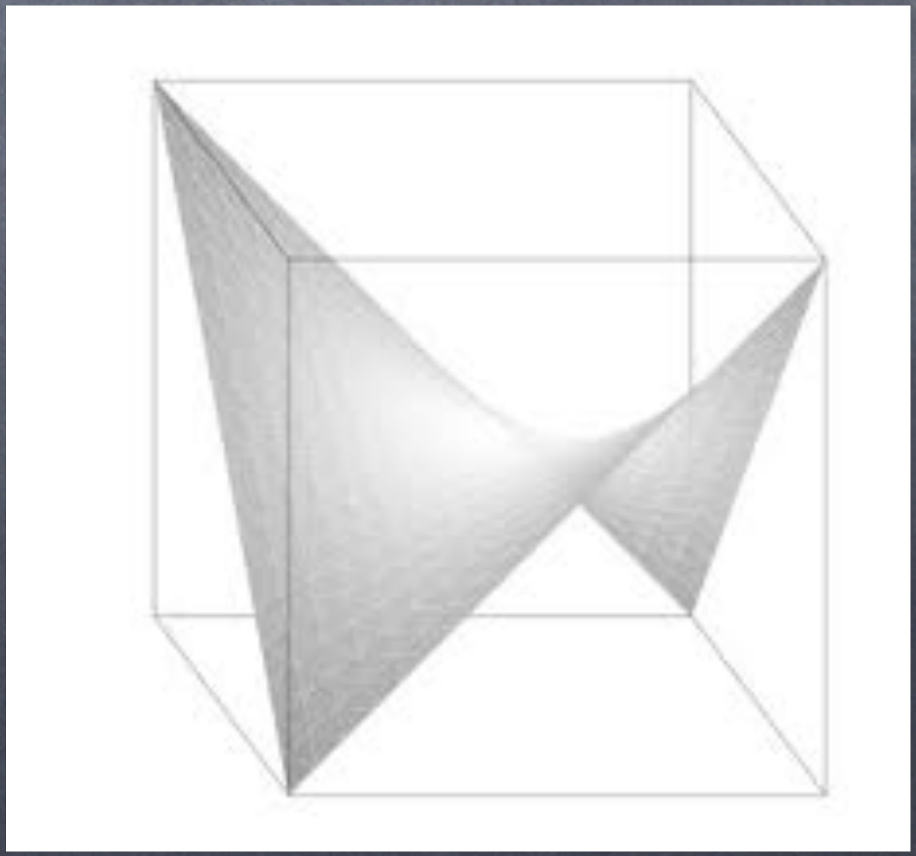
Is there a least-area surface whose boundary is a given simple closed curve?

More generally, Minimal Surfaces

Soap films

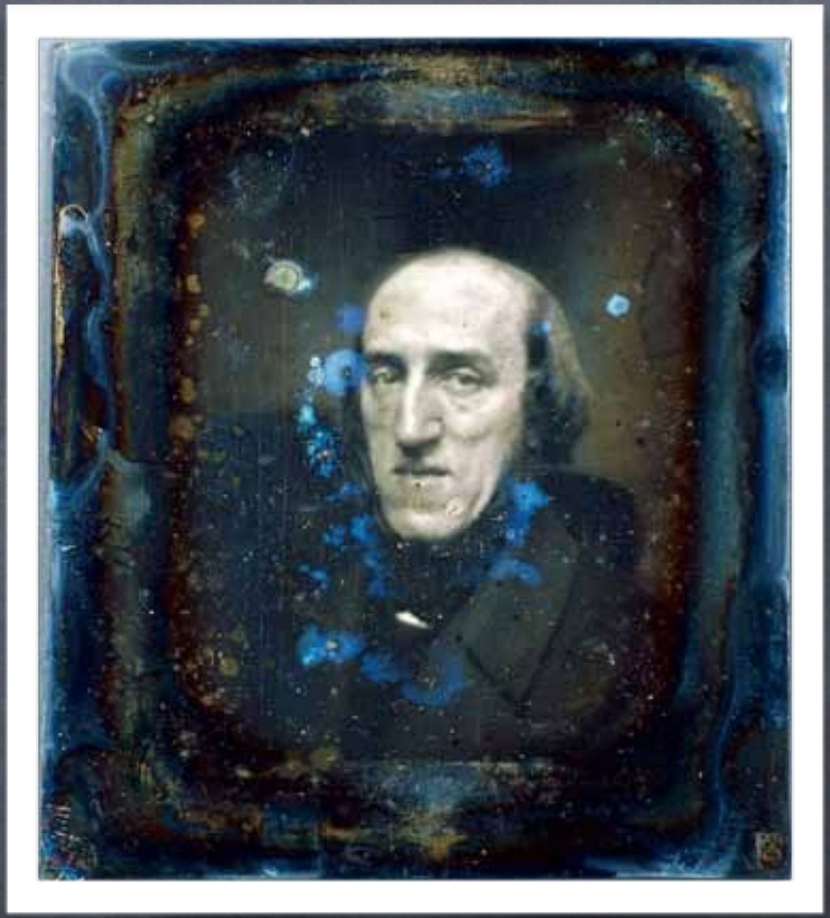
Soap films

Soap films always adopt the shape which minimizes their elastic energy, and therefore their area. (Minimal Surfaces)



Soap films

In the 1800s, the physicist J. Plateau (problem raised by Lagrange) conducted classical experiments in this area, dipping wires bent into assorted shapes in tubs of soapy water. Plateau concluded that the soap films that formed were always minimal surfaces. Plateau hypothesized that for any given closed curve, you can always produce a minimal surface with the same boundary.



Soap films

Plateau's Problem: "Is there a least-area surface whose boundary is a given simple closed curve?"

The Plateau's Problem was solved by J. Douglas and (independently) T. Radó in 1930.

For its solution to the Plateau's Problem; Douglas (right) was awarded with the Fields Medal

Surfaces on Riemannian Geometry

Basics on Riemannian Geometry

(M^3, g) three-dimensional Riemannian manifold

g measures the length of vector fields on the tangent space TM^3

∇ tells how to differentiate vector fields

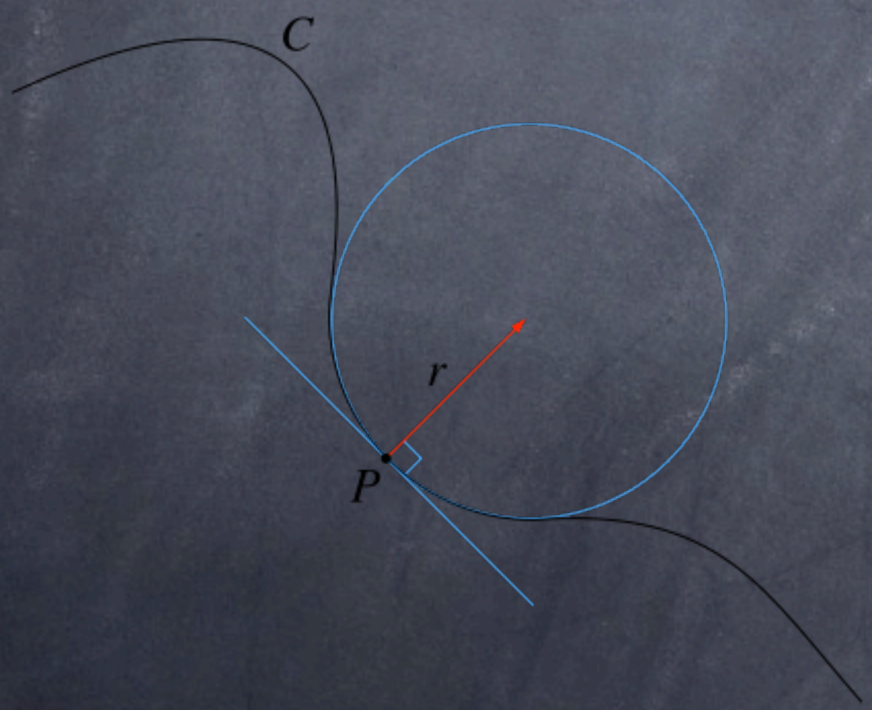
R measures the curvature at each point

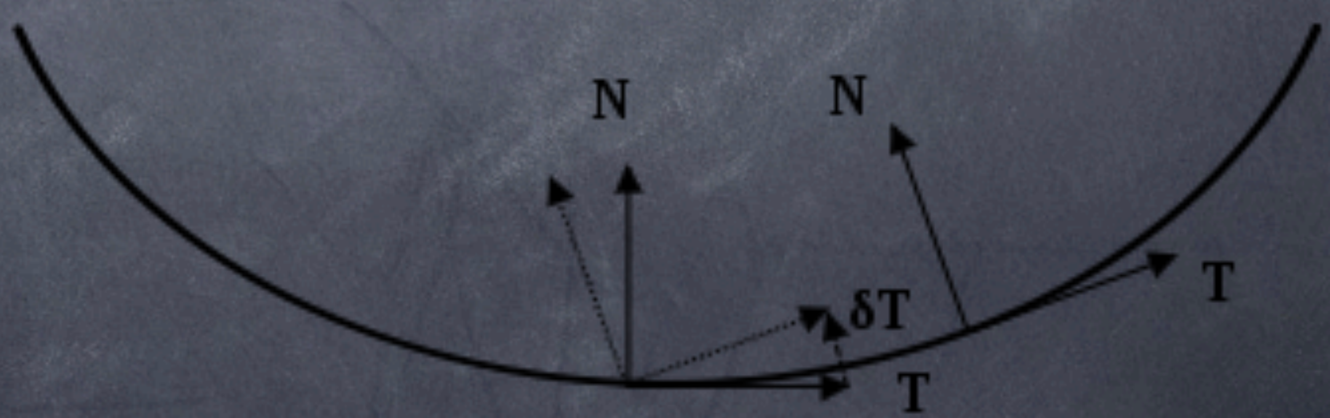
$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Basics on Riemannian Geometry

Notion of curvature

Curves: it is the variation of the normal vector, i.e., the acceleration is related to the normal by the curvature.

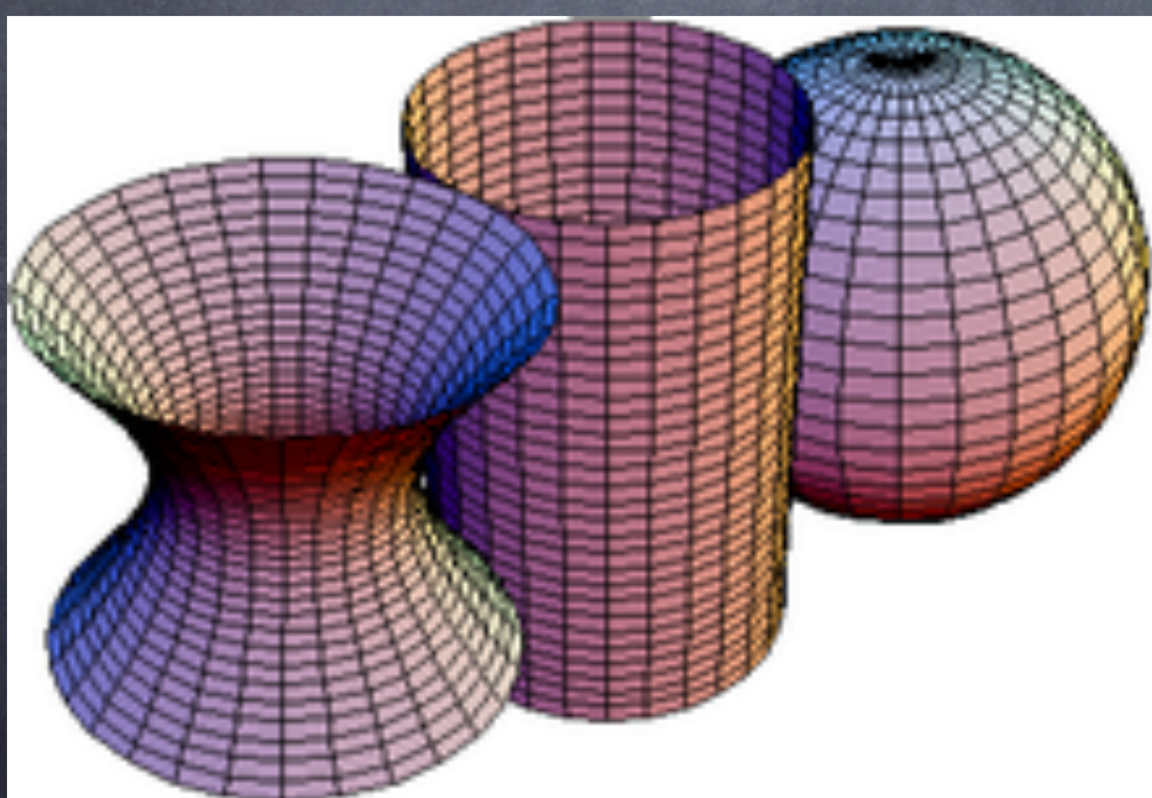


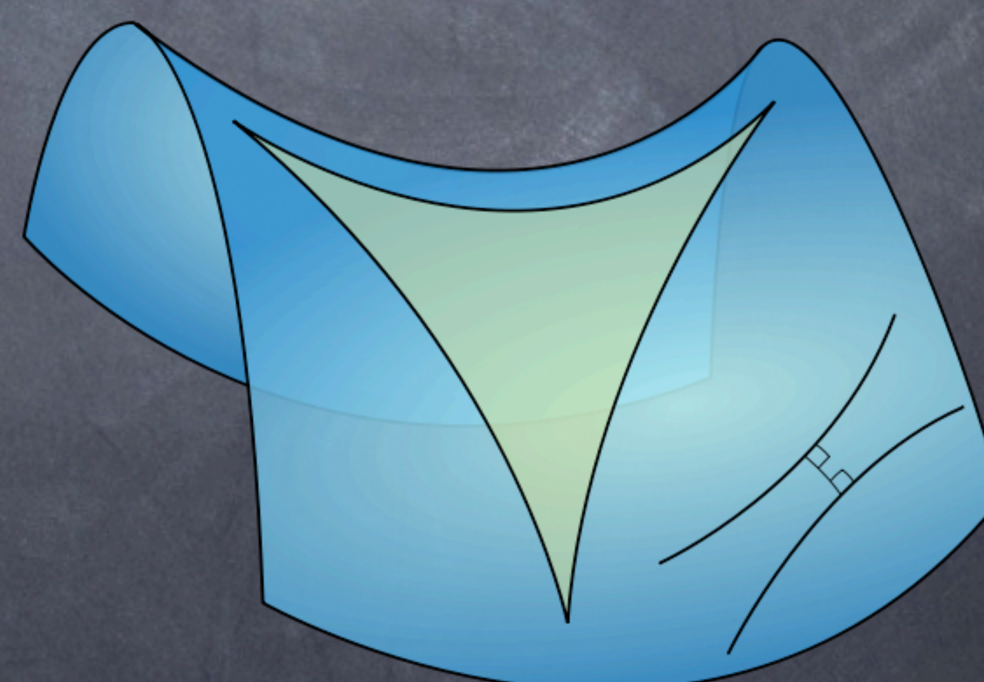


Basics on Riemannian Geometry

Notion of curvature

Surfaces: The Riemann Curvature Tensor R is completely determined by a function, the Gaussian Curvature K. K measures the deviation from being flat.





Basics on Riemannian Geometry

Notion of curvature

3-manifold: The Riemann Curvature Tensor R measures the extent to which the metric tensor is not locally isometric to a Euclidean space.

- Sectional Curvature K_s:

$$K_s(e_i,e_j) := g(R(e_i,e_j)e_i,e_j)$$

- Ricci Curvature Ric:

$$Ric := K_s(e_i,e_j)\, dx_i dx_j$$

- Scalar Curvature S:

$$S := Trace(Ric)$$

Basics on Riemannian Geometry

(M^3,g) three-dimensional Riemannian manifold. Let $\Sigma \subset M^3$ be a complete and oriented surface.

- Intrinsic Geometry: g induces a Riemannian metric on Σ.

Gaussian Curvature K

- Extrinsic Geometry: N induces a self-adjoint operator on the tangent space $T\Sigma$.

Principal Curvatures k_1 and k_2

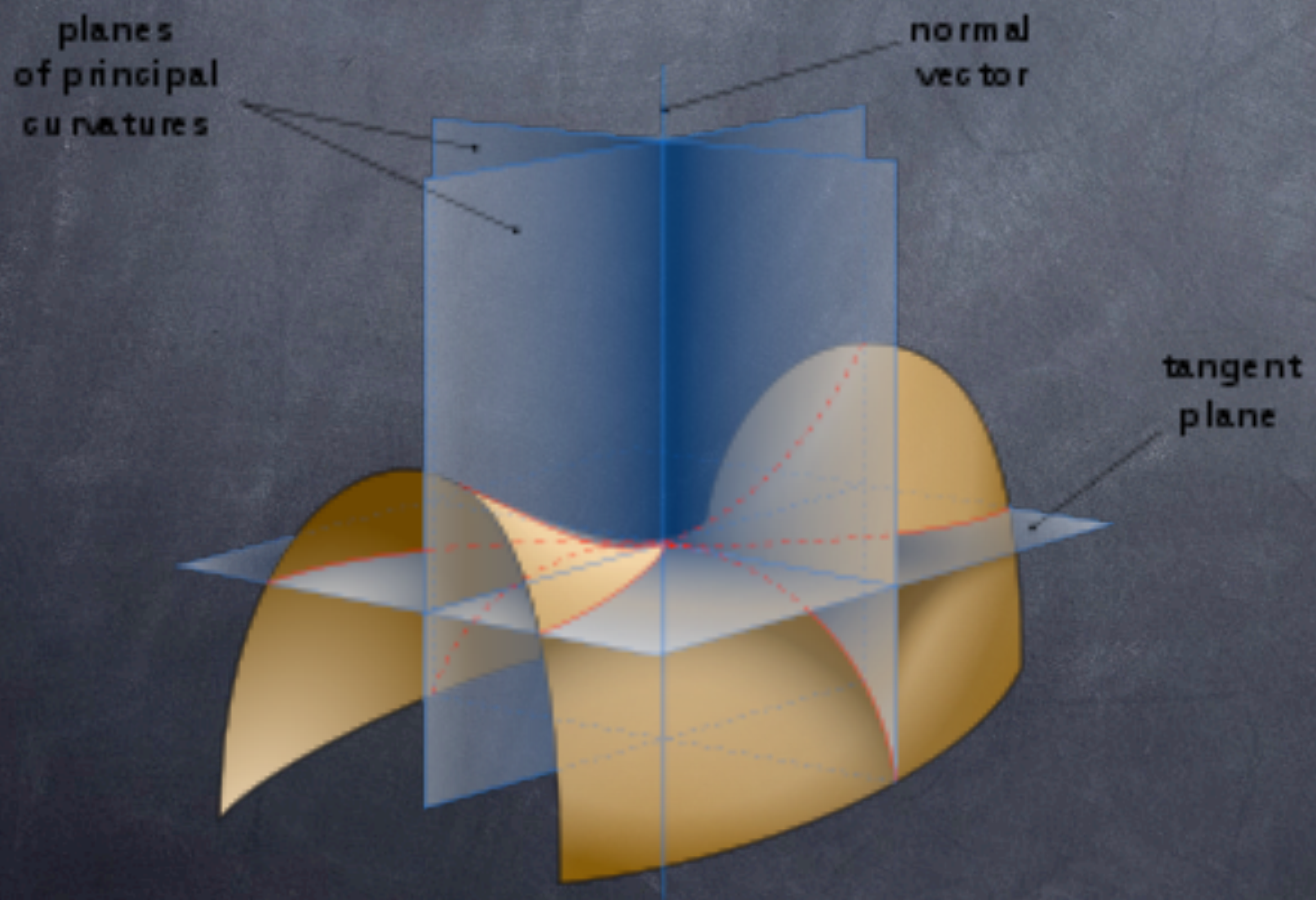
Extrinsic Curvature K_e

Mean Curvature H

Basics on Riemannian Geometry

(M^3, g) three-dimensional Riemannian manifold. Let $\Sigma \subset M^3$ be a complete and oriented surface.

Intrinsic & Extrinsic Geometry:



Gauss Equation

$K = K_e + K_s$

Minimal Surfaces

Minimal surfaces

$\Sigma \subset M^3$ (oriented and connected) is a minimal surface iff critical point of the area functional:

F:= Area, i.e. $$\frac{d}{dt}\bigg|_{t=0} F(t) = \int_\Sigma f\, H = 0$$

for all compactly supported normal variations:

$$\Sigma(t) := \{\exp_p(tf(p)N(p)) : p \in \Sigma\}, \qquad |t| < \varepsilon$$

N := normal along Σ

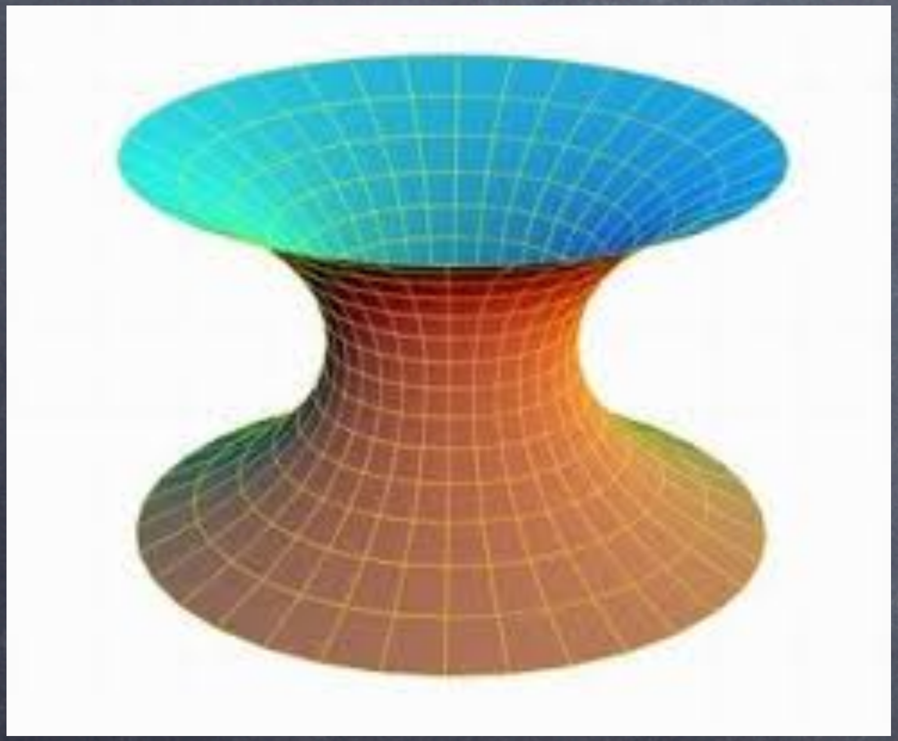
exp := exponential map on M^3

f := piecewise smooth compact support

Examples

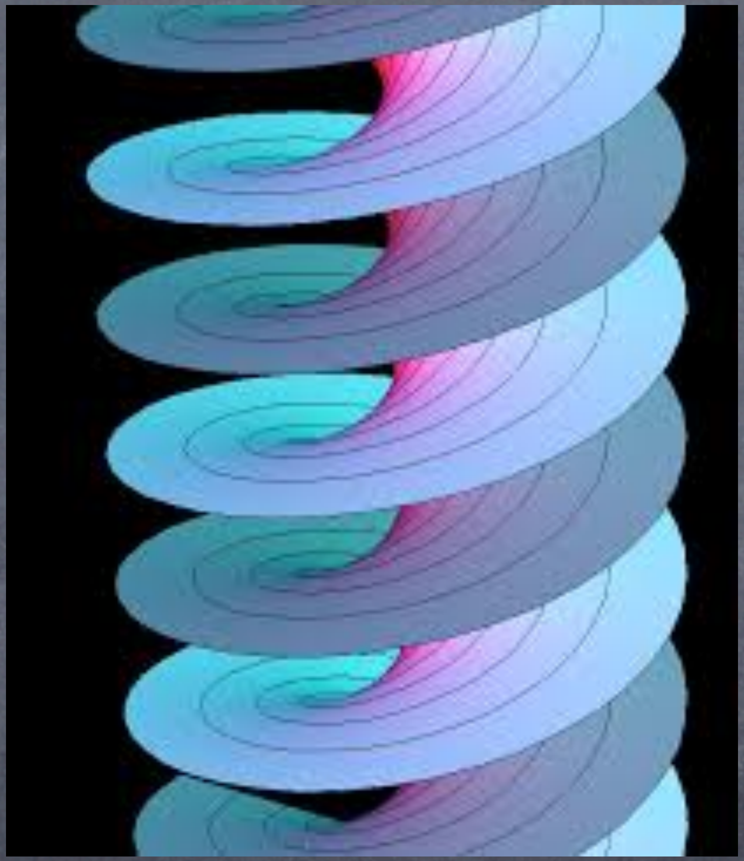
The plane: Lagrange 1762

Catenoid



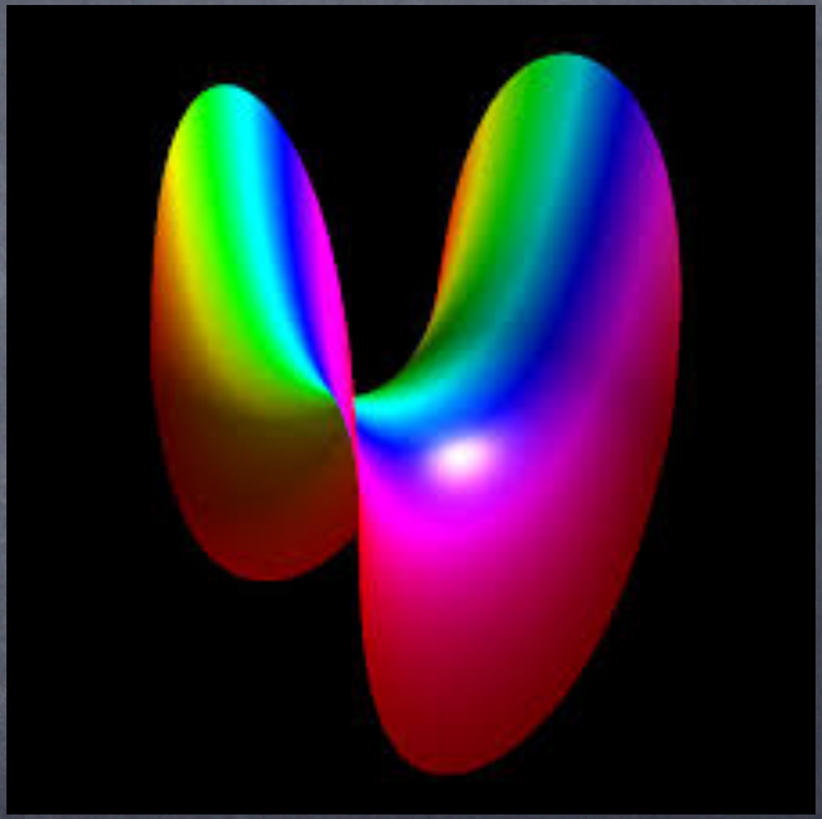
Euler 1744

Helicoid



Euler 1744/
Meusnier 1776

Scherk Surface



1830

History

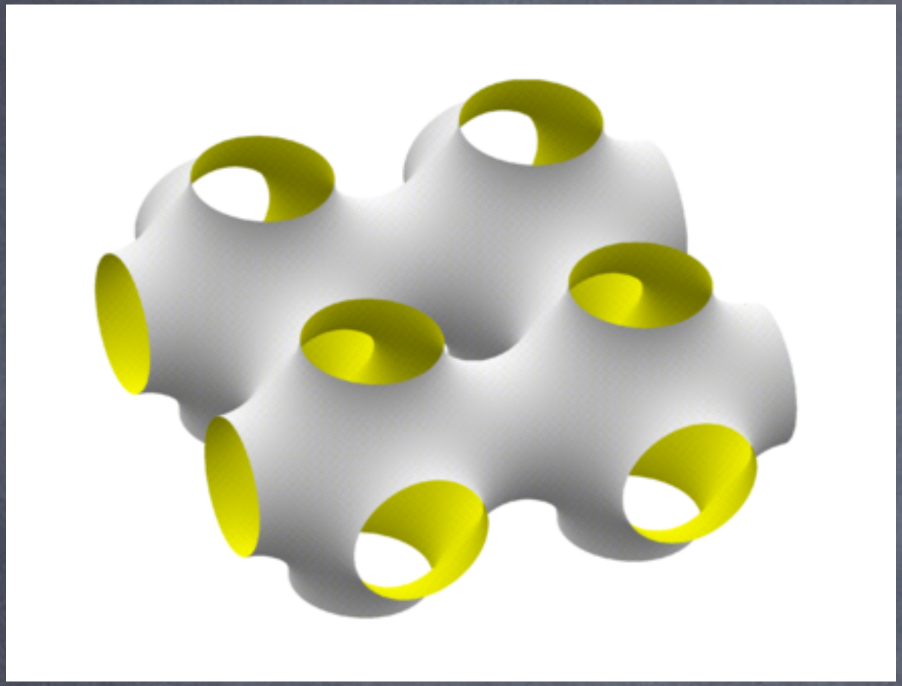
Schwarz (1865): solution to the Plateau Problem for a regular quadrilateral

Schwarz (1867): solution to the Plateau Problem for a general quadrilateral; allowing the construction of periodic minimal surfaces

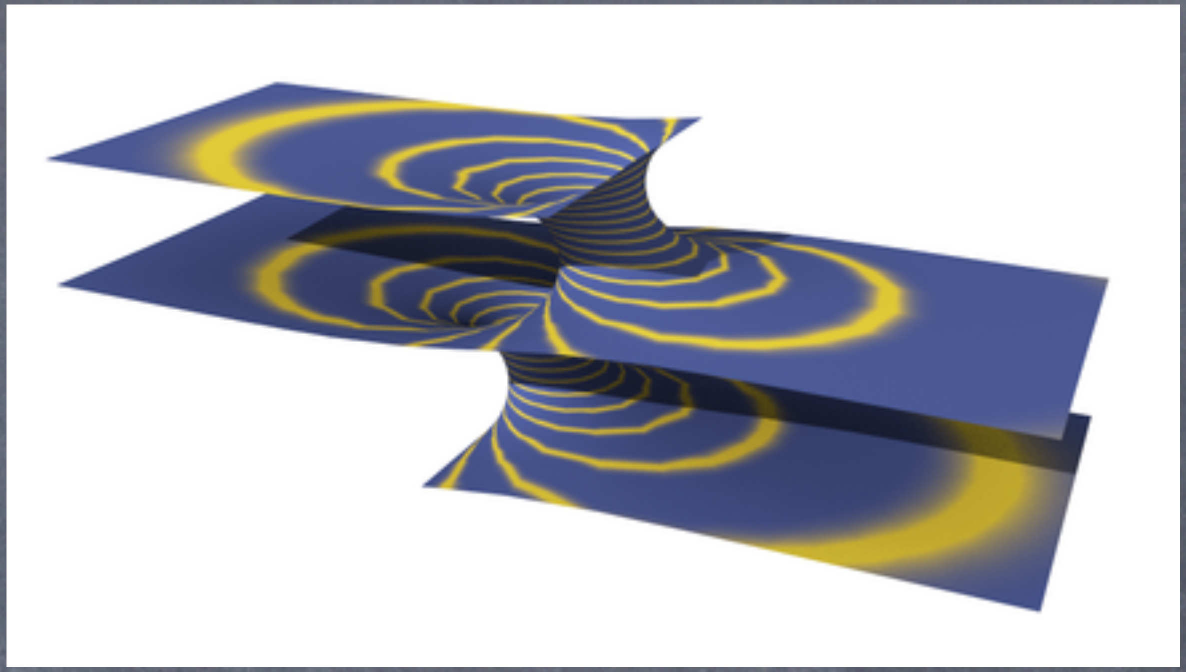
Enneper-Weierstrass Representation (1863): linking minimal surfaces to complex analysis and harmonic functions

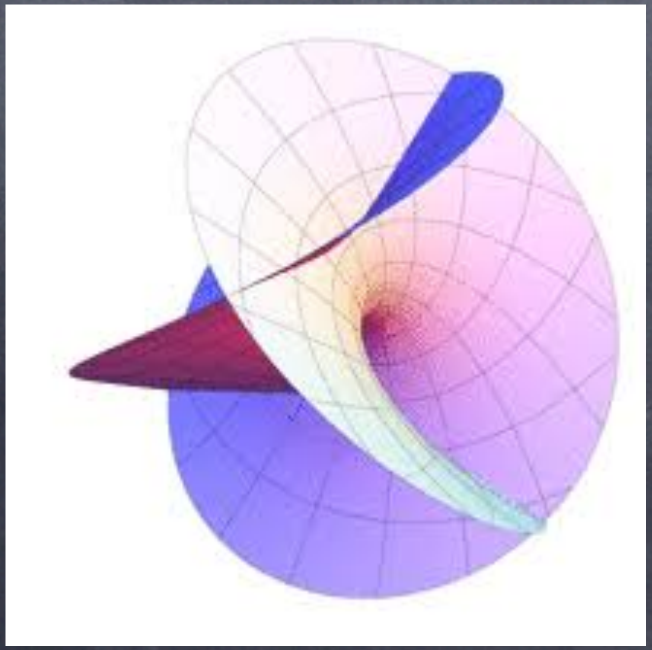
Examples

P-Schwarz (1880s)



Riemann example (1867)

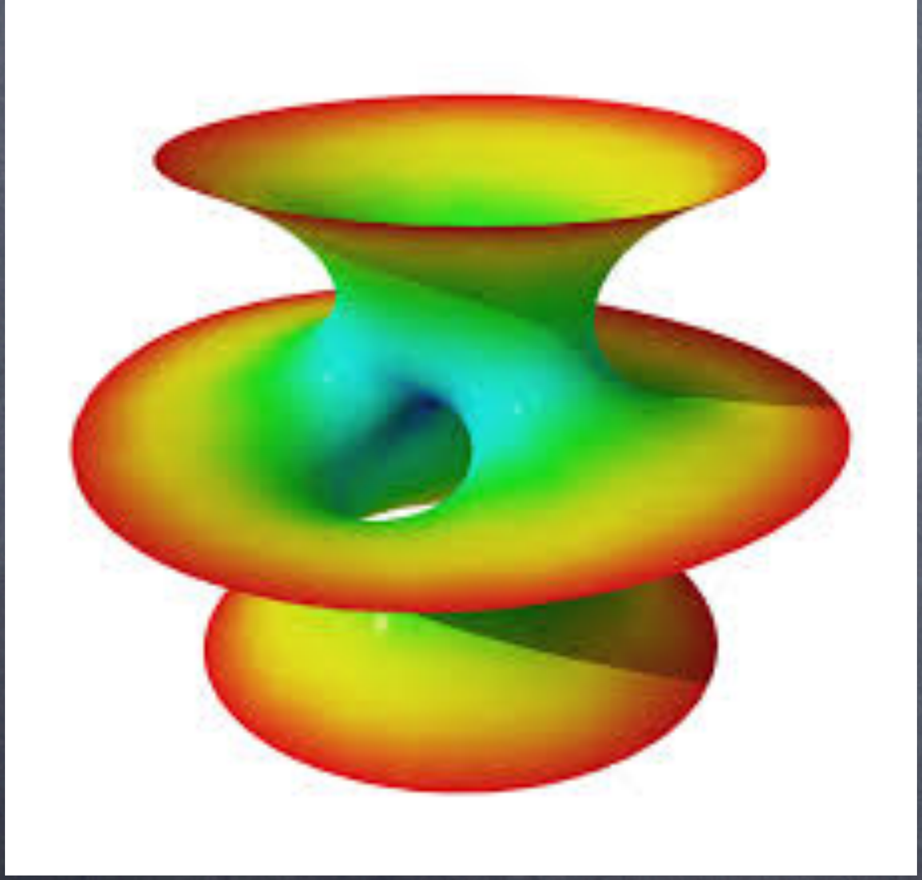




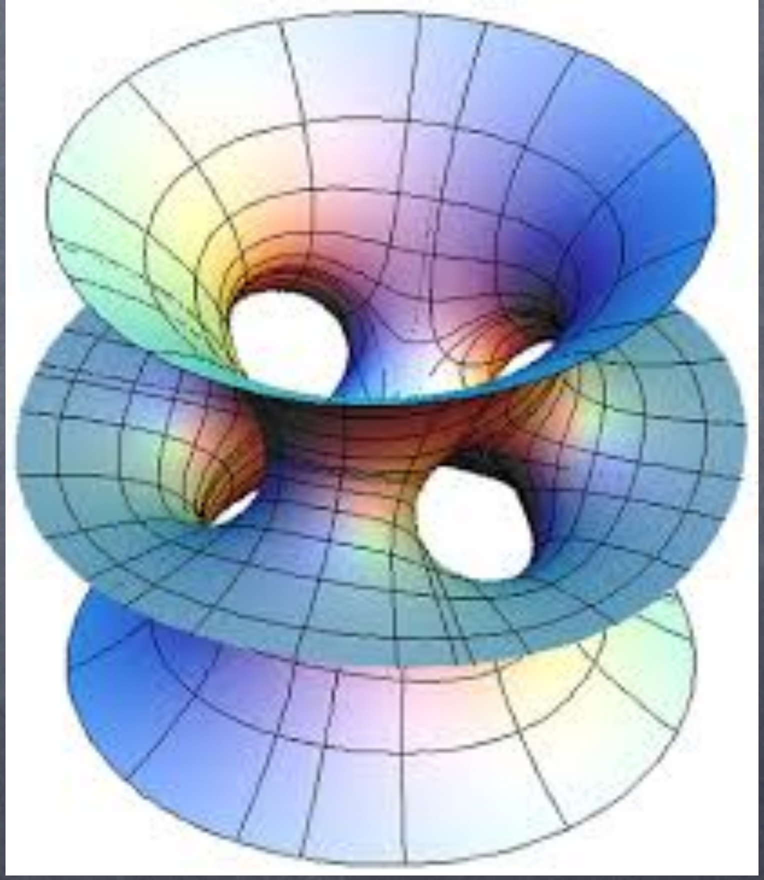
Enneper Surface (1864)

Examples

For a long time, some geometers supported the conjecture that no other embedded examples of finite topology would exist. More than 100 years later......



Costa (1982)



Costa-Hoffman-Meeks

Classification of Minimal Surfaces

Stable surfaces

Among complete minimal surfaces in M^3, the subclass of stable minimal surfaces is the first one to be understood and described. $\Sigma \subset M^3$ stable iff

$$\left.\frac{d^2}{dt^2}\right|_{t=0} F(t) \geq 0 \qquad \text{("minimizer")}$$

Schoen-Yau (H=0):

$$\int_\Sigma f^2 |A|^2 + \int_\Sigma f^2 \mathrm{Ric}(N,N) \leq \int_\Sigma |\nabla f|^2, \quad \forall f \in H_0^{1,2}(\Sigma)$$

Stable surfaces

One writes the stability inequality in the form

$$\frac{d^2}{dt^2}\bigg|_{t=0} F(t) = -\int_\Sigma fLf \geq 0$$

where L is the Shrödinger-type operator

$$L := \Delta + |A|^2 + Ric(N,N) = \Delta - K + (4H^2 - K_e + S)$$

L is known as the linearized operator of the mean curvature, i.e.,

$$\frac{d}{dt}\bigg|_{t=0} H(t) = Lf$$

Schrödinger & stability

Let Σ be a Riem. surface and let L be the Schrödinger operator given by:

$$L := \Delta + q, \; q \text{ smooth on } \Sigma.$$

These type of equations (time-independent version) were used in physics, by E. Schrödinger in 1926, to describe the quantum state of a physical system. Later, M. Born interpreted solutions to the Schrödinger equation as a quantity related to the probability amplitude.

Schrödinger & stability

Let Σ be a Riem. surface and let L be the Schrödinger operator given by:

$$L := \Delta + q, \; q \text{ smooth on } \Sigma.$$

The Index Form is given by:

$$I(f) := -\int_\Sigma f\, Lf, \qquad \forall f \in H_0^{1,2}(\Sigma)$$

I defines a quadratic form in a infinite dimensional space, therefore (Classical Elliptic Theory) we have:

$$-\infty < l_1 < l_2 < \ldots.$$

Schrödinger & stability

Let Σ be a Riem. surface and let L be the Schrödinger operator given by:

$L := \Delta + q$, q smooth on Σ.

The Index Form is given by:

$$I(f) := -\int_\Sigma f\,Lf, \qquad \forall f \in H_0^{1,2}(\Sigma)$$

L is stable iff $0 \le l_1$ iff $0 \le I(f), \forall f \in H_0^{1,2}(\Sigma)$

Schrödinger & stability

(A) Analytically: In terms of L, stability means that $-L$ is non-negative (in short $-L \leq 0$), i.e., all its eigenvalues are non-negatives. Moreover, Σ has finite index if $-L$ has only finitely many eigenvalues.

(B) Geometrically: Σ has finite index if there is only a finite dimensional space of normal variations which strictly decreases the area.

Stable surfaces

Do Carmo & Peng ('79): A complete, oriented stable minimal surface in the Euclidean space is a plane.

Schoen & Yau ('79): Let $\Sigma \subset M^3$ be a compact, oriented stable minimal surface (M^3 has nonnegative scalar curvature). Then Σ is a sphere or a flat torus.

Schoen & Yau ('79): Let M^3 be a compact orientable 3-manifold with nonnegative scalar curvature. If M^3 contains an incompressible compact orientable surface Σ with genus greater than or equal to 1, then M^3 is flat.

Stable surfaces

Fischer-Colbrie & Schoen ('80): Let $\Sigma \subset M^3$ be a complete, oriented stable minimal surface (M^3 has nonnegative scalar curvature). then Σ is conformally equivalent either to the plane or to the cylinder. Moreover, if Σ is conf. a cylinder and $\int_\Sigma |K| < +\infty$, then Σ is totally geodesic and S vanishes along Σ

M. Reiris ('10) and E. ('10): Under the above hypothesis. If Σ is conf. a cylinder, then Σ is flat, totally geodesic and S vanishes along Σ.

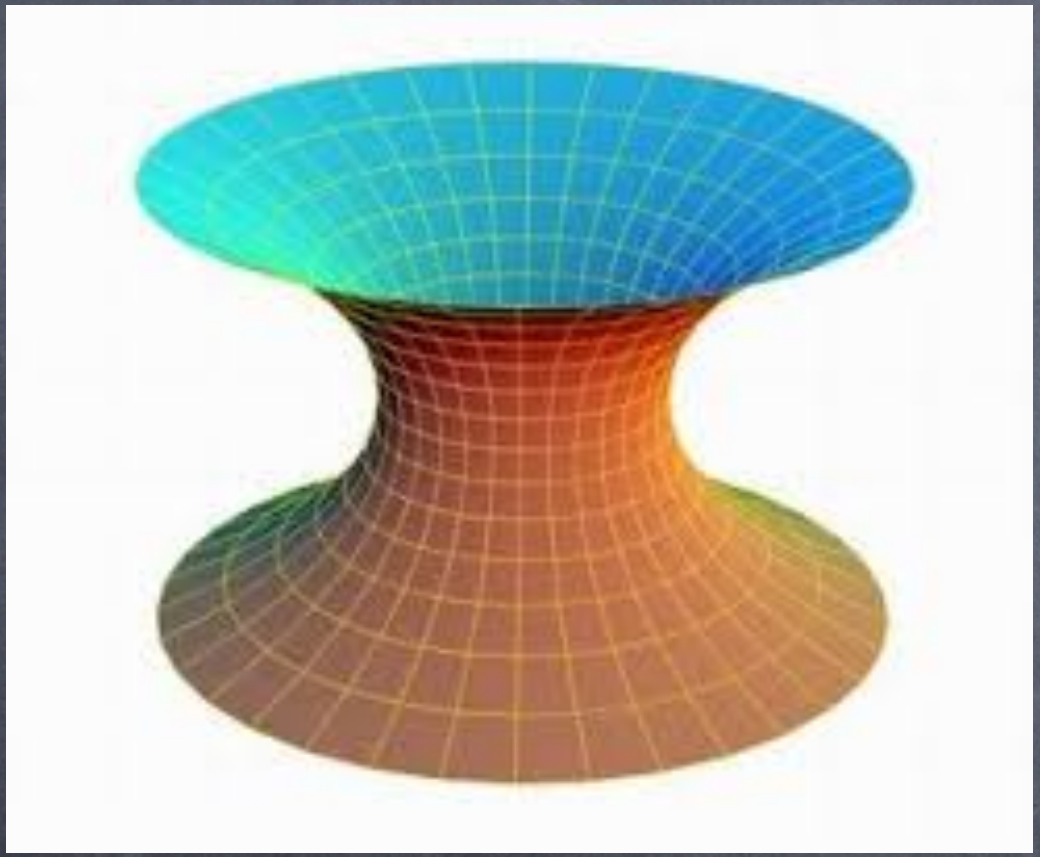
Finite Index

Fischer-Colbrie (80s): Let $\Sigma \subset \mathbb{R}^3$ be a complete, oriented minimal surface. Σ has finite index iff $\int K < +\infty$; i.e., Σ has finite total curvature

Osserman/Schoen/White: Finite total curvature implies that Σ is proper and conformally equivalent to a compact surface punctured at a finite number of points. Moreover, each embedded end (punctures), is asymptotically a plane or a half-catenoid.

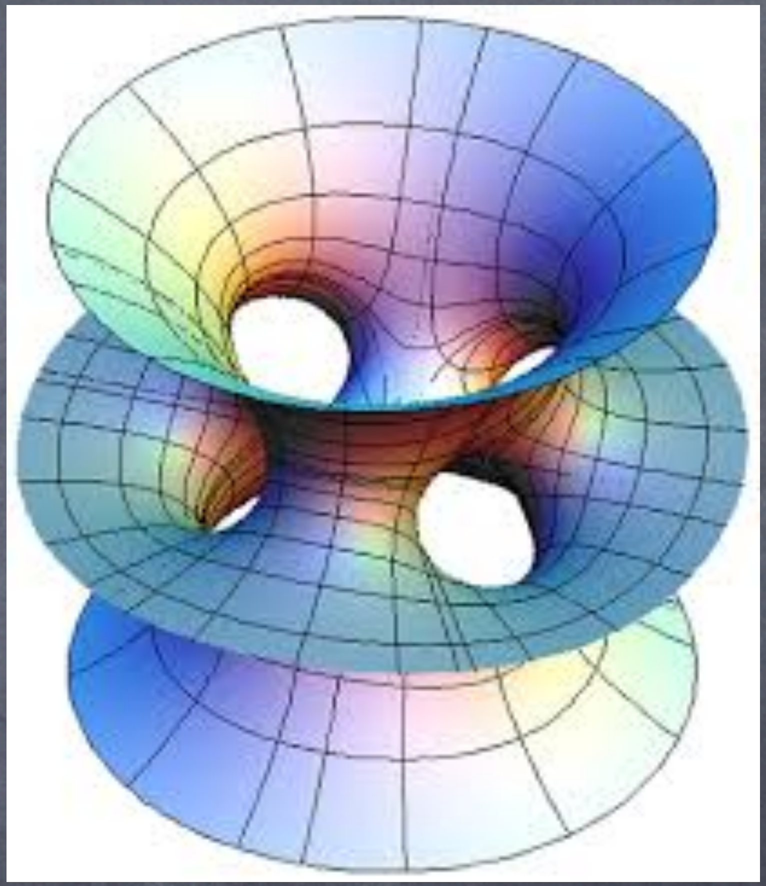
Finite Index

Index 1



Catenoid

Index 2g+3



Costa-Hoffman-Meeks

Finite Index

Fischer-Colbrie-Schoen (80)/do Carmo-Peng (79)/ Pogorelov (81): The only complete orientable stable minimal surface is the plane

Schoen (83): The plane is the unique complete orientable finite index and one ended minimal surface

Ros (2006): The only complete stable minimal surface is the plane

López-Ros (89): The catenoid and Enneper's surface are the unique complete orientable index one minimal surfaces

Finite Index

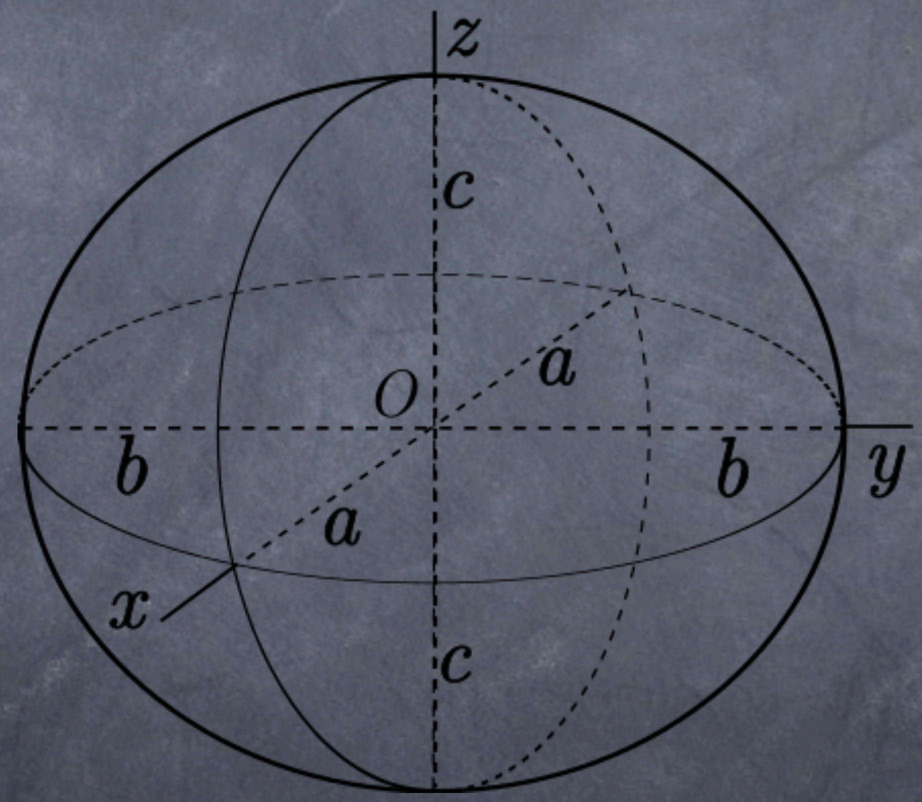
Schoen (83): The catenoid is the unique complete embedded orientable finite index and two ended minimal surface

López-Ros (89): The plane and catenoid are the unique complete embedded orientable finite index minimal surfaces with genus zero

Chodosh-Maximo (2017): There are no complete embedded orientable minimal surfaces with index 2 or 3

Yau's conjecture

Poincaré (1905): Conjectured that every smooth surface topologically equivalent to the sphere contains three simple closed geodesics



Question: infinite number??

Yau's conjecture

Birkhoff (1917): The first to use a Mountain Pass type argument to construct a simple closed geodesics in any 2-sphere (min-max method)

Ballmann ('78): Proved the Poincaré's Conjecture

Franks ('92): Infinitely many (immersed) in 2-spheres using Dynamical Systems

Bangert ('93): Infinitely many (immersed) in 2-spheres using min-max methods

Yau's conjecture

Yau's conjecture (82): A closed Riemannian three-manifold has an infinite number of smooth closed immersed minimal surfaces.

Yau's conjecture

Almgrem-Pitts (81): Any closed Riemannian manifold of dimension at least 3 and less than 7, there exists a smoothly embedded closed minimal hypersurface (min-max theory)

Marques-Neves (17): Settled Yau's Conjecture when the Ricci curvature is positive and dimension less than 7 (min-max theory)

Marques-Neves-Irie (18): Settled Yau's Conjecture for generic metrics and dimension less than 7 (min-max theory and Dynamical Systems)

Yau's conjecture

A. Song (18): Settled Yau's Conjecture in dimension less than 7

Osserman's conjecture

Bernstein (1917): The plane is the only entire graph over a plane in $\mathbb{R}^3$

Osserman's Conjecture (80s): Asked the question about whether the plane and the helicoid were the only embedded, simply-connected, complete minimal surfaces. He described this question as potentially the most beautiful extension and explanation of Bernstein's Theorem.

Osserman's conjecture

Colding-Minicozzi (04): A connected, complete, embedded minimal surface of finite topology in R^3 is properly embedded.

They used the One Sided Curvature Estimates by Colding-Minicozzi

Osserman's conjecture

Meeks-Rosenberg (2006): A complete, embedded, simply-connected minimal surface in R^3 is a plane or a helicoid.

They used the One Sided Curvature Estimates by Colding-Minicozzi (2004)

Classification Planar domains

Meeks-Pérez-Ros (2006): Up to scaling and rigid motion, any connected, properly embedded, minimal planar domain in R^3 is a plane, a helicoid, a catenoid or one of the Riemann minimal examples. In particular, for every such surface there exists a foliation of R^3 by parallel planes, each of which intersects the surface transversely in a connected curve which is a circle or a line.

Black holes

Special Relativity

"Reflections of this type made it clear to me as long ago as shortly after 1900, i.e., shortly after Planck's trailblazing work, that neither mechanics nor electrodynamics could (except in limiting cases) claim exact validity. Gradually I despaired of the possibility of discovering the true laws by means of constructive efforts based on known facts. The longer and the more desperately I tried, the more I came to the conviction that only the discovery of a universal formal principle could lead us to assured results... How, then, could such a universal principle be found?"

Albert Einstein

Special Relativity

In 1905, Einstein proposed (after considerable and independent contributions of Lorentz, Poincaré and others) the Special Relativity Theory:

- Principle of Relativity: All uniform motion is relative, and there is no absolute and well-defined state of rest, for all the laws of physics.

- Principle of Invariant Light Speed: The speed of light in empty space is the same for all inertial observers, regardless of the state of motion of the source.

General Relativity

In 1915, Einstein included gravity through the Equivalence Principle:

"A little reflection will show that the law of the equality of the inertial and gravitational mass is equivalent to the assertion that the acceleration imparted to a body by a gravitational field is independent of the nature of the body. For Newton's equation of motion in a gravitational field, written out in full, it is:

(Inertial mass) (Acceleration) = (Intensity of the gravitational field) (Gravitational mass).

It is only when there is numerical equality between the inertial and gravitational mass that the acceleration is independent of the nature of the body."

A. Einstein

General Relativity

- In 1915, A. Einstein included gravity through the Equivalence Principle, this relates:

Mass <---> Acceleration <---> Curvature

Therefore,

Gravity <---> Geometry

- Last step, Einstein needed to include a Principle of least action for deriving the equations of motion. This was done using the Einstein-Hilbert action

Math of General Relativity

Let (M,g) a 4-dimensional Lorentzian manifold, also called spacetime.

The Einstein field equations are given by:

$$2T := 2\,\mathrm{Ric} - S\,g$$

- T := stress-energy tensor.

It measures the local mass and momentum

- G := 2 Ric - S g, the Hilbert-Einstein tensor

Math of General Relativity

Let $(\underline{M},\underline{g})$ a 4-dimensional Lorentzian manifold, also called spacetime.

Let $M \subset \underline{M}$ be a 3-dimensional (embedded and orientable) spacelike hypersurface.

- g induced metric on M
- n normal along M <--> B second fund. form

(M,g,B) is an initial data set

Math of General Relativity

Given an initial data set (M,g,B) in $(\underline{M},\underline{g})$, we can express T in terms of the initial data set:

- $2\rho := S - |B|^2 + (Tr(B))^2$ local mass
- $J := div(B - Tr(B)g)$ local momentum

"Dominant Energy Condition"

- T is timelike or null <---> $|J| \leq \rho$

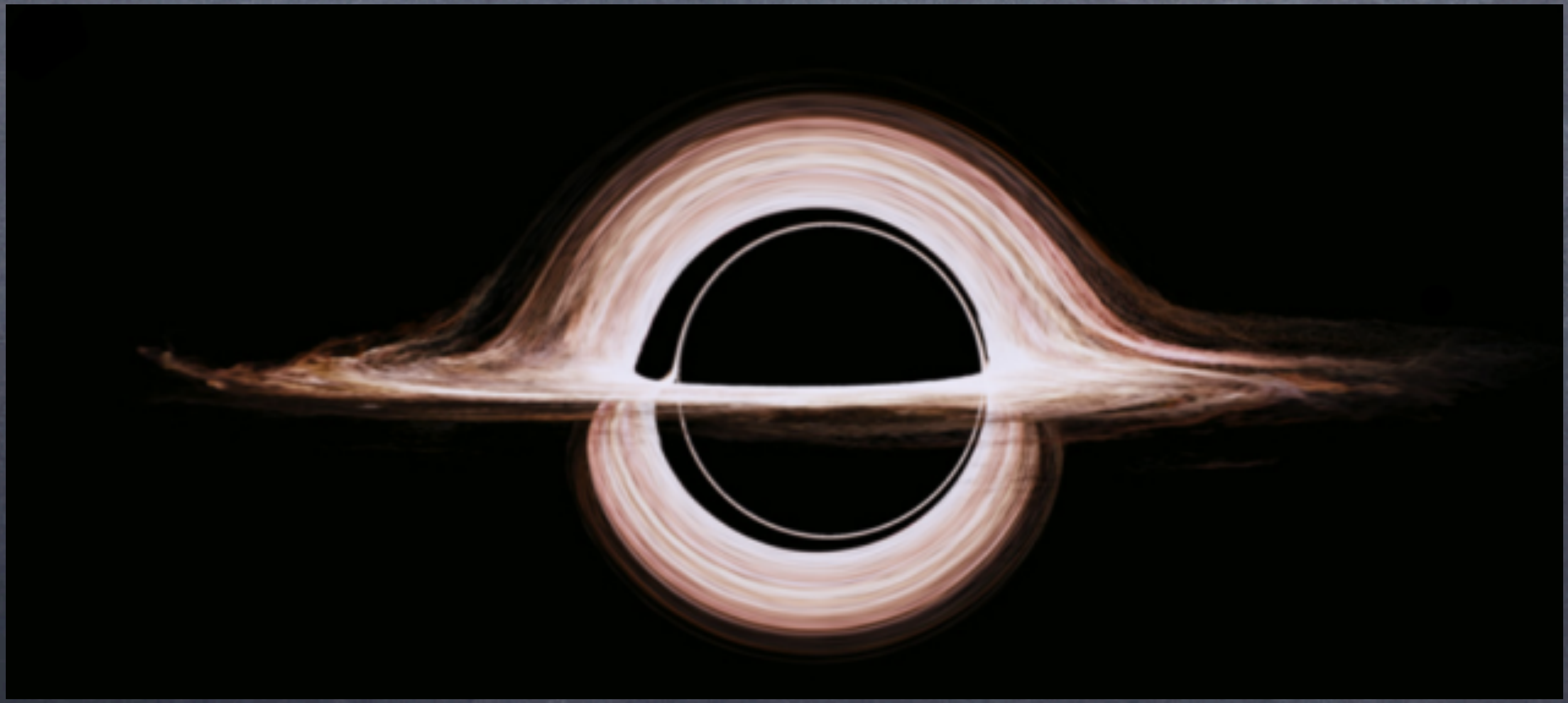
Black holes

- One of the most fascinating objects in General Relativity are Black Holes.
- Loosely speaking, black holes are singular solutions to the Einstein Field Equations and they are created by gravitational collapse.

Black holes

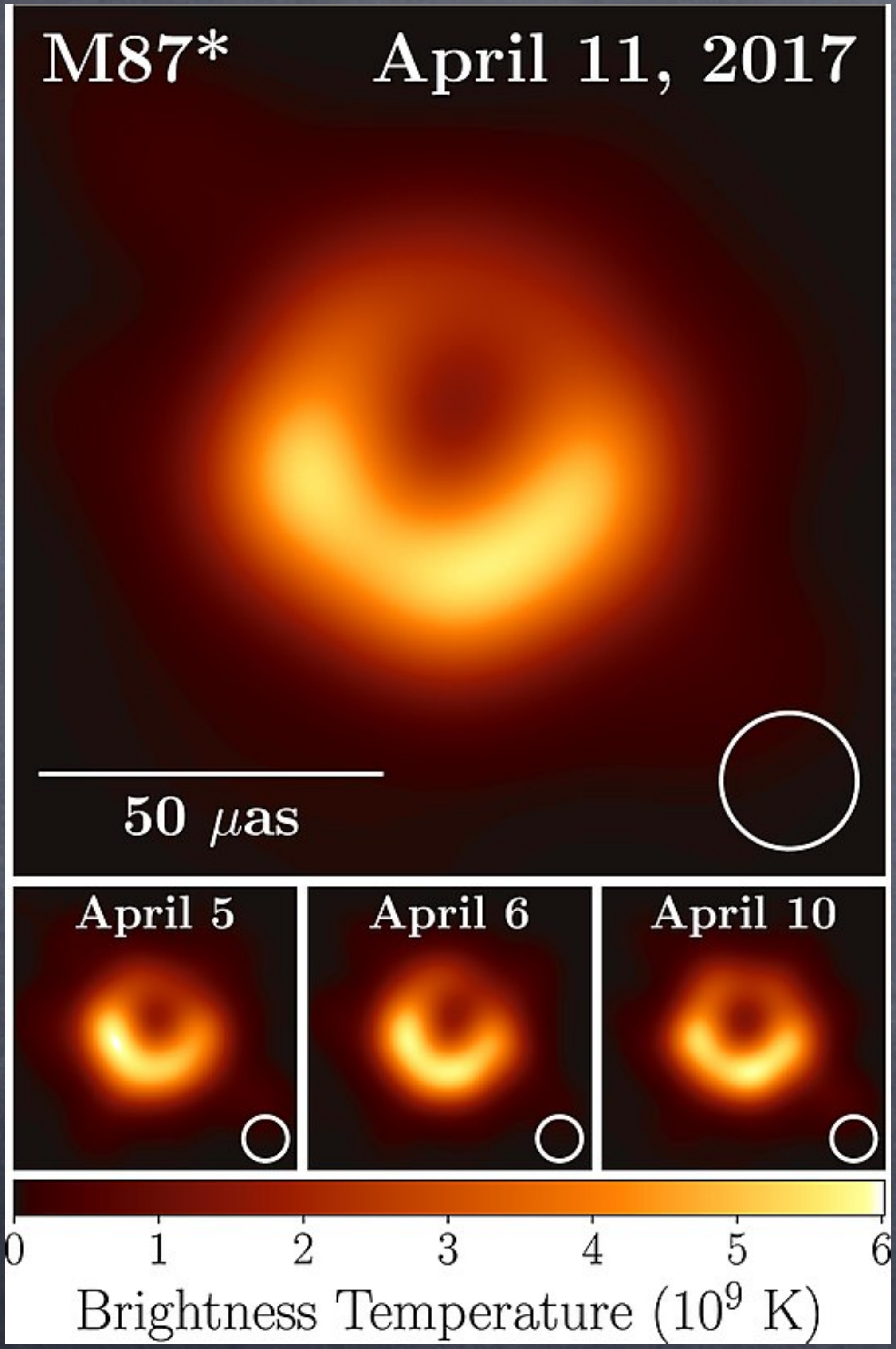
- The Penrose-Hawking Singularity Theorems attempt to answer when gravitation produces singularities (naked or not)

- Cosmic Censorship Hypothesis (Penrose) asserts that no naked singularity exists



(From "Interstelar")

Black holes



First image of a Black Hole !!!!!

Black holes

- Black hole event horizon is the boundary in spacetime beyond which light cannot scape black hole's gravitational force, that is, events in the interior of the event horizon cannot affect an outside observer.

- From a mathematical point of view, Marginally Outer trapped surfaces MOTS are widely considered as a good quasi-local replacement of the event horizon of a black hole, being such surfaces the natural analog to minimal surfaces in General Relativity.

Math of Black holes

Let $\Sigma \subset (M, g, B) \subset (\underline{M}, \underline{g})$ be a (orientable) compact surface in an initial data set.

n := normal along M in $\underline{M}$; $\underline{g}(n,n) = -1$

N := normal along Σ in M; $\underline{g}(n,n) = 1$

$v := n+N$; $\underline{g}(v,v) = 0$. Future directed null vector

$C := A+B$; Second Fundamental Form w.r.t. v

$$h = \mathrm{Trace}_{\Sigma}(C) = H + \mathrm{Trace}_{\Sigma}(B)$$

Positive Null Expansion

Math of Black holes

Let $\Sigma \subset (M,g,B) \subset (\underline{M},\underline{g})$ is a **Marginally Outer Trapped Surface**, MOTS, iff

$$h = H + \text{Trace}(B) = 0$$

Let $\Sigma \subset (M,g,B) \subset (\underline{M},\underline{g})$ is a **Apparent Horizon** iff

i) Σ is a MOTS
ii) There are no outer trapped surfaces outside Σ

"Heuristically, Σ is the outer limit of outer trapped surfaces"

When **Trace(B)=0**, Σ is a **minimal surface** !!!

Hawking's Theorem

Hawking ('72): The cross sections of the event horizon in 4-dimensional asymptotically flat stationary black hole spacetimes obeying the dominant energy condition are topologically 2-spheres.

Positive Mass Theorem

Assuming the Dominant Energy Condition, the mass of an asymptotically flat spacetime is non-negative; furthermore, the mass is zero only for Minkowski spacetime

Schoen-Yau (79): Using minimal surfaces up to dimension 7

Witten (81): Assuming the manifold is spin; inspired by Positive Energy Theorems in Supergravity

Schoen-Yau ('17): Using minimal surfaces and no restriction on the dimension

Penrose Inequality

The Penrose inequality estimates the mass of a spacetime in terms of the total area of its black holes and is a generalization of the positive mass theorem.

$$M \geq \sqrt{\frac{A}{16\pi}}$$

Riemannian Case

Huisen-Illmanen (97): Using Inverse Mean Curvature Flow

Bray (99): Using a Conformal Flow of metrics

THANKS!!!