Estabilización de sistemas conmutados

Juan Bosco García Gutiérrez

V Jornadas Doctorales del Programa de Matemáticas

Universidad de Cádiz

Martes, 19 de noviembre de 2019

Outline of this presentation

1. Switched systems
2. Invariant sets for third order switched systems
3. A stabilization condition for third order switched systems
4. A new method for stabilization

Outline of this section

1. **Switched systems**
   - Definition of switched system
   - Solution of a switched system
   - Example of second order switched system
   - Problems on switched systems

Definition of switched system

Switched system

A **switched system** is given by a family of systems

$$\dot{x} = f_\sigma(x),$$

where $f_j : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field with index $j \in J$ which is called **subsystem**, the variable x is the **state variable** and the number n is the **order** of the system.

$\sigma : \mathbb{R}_+ \to J$ is a switching law.

$$\dot{x}(t) = f_{\sigma(t)}(x(t)).$$

$\sigma : \mathbb{R}^n \to J$ is a feedback switching law.

$$\dot{x}(t) = f_{\sigma(x(t))}(x(t)).$$

Solution of a switched system

$$\begin{cases} \dot{x}(t) = f_{\sigma(t)}(x(t)), \;\; t \geq 0, \\ x(0) = x_0, \end{cases}$$

Solution of a switched system

$$\begin{cases} \dot{x}(t) = f_{\sigma(t)}(x(t)), \ \ t \geq 0, \\ x(0) = x_0, \end{cases}$$

Denote the solution $\varphi(t; x_0, \sigma)$.

Solution of a switched system

$$\begin{cases} \dot{x}(t) = f_{\sigma(t)}(x(t)), \ \ t \geq 0, \\ x(0) = x_0, \end{cases}$$

Denote the solution $\varphi(t; x_0, \sigma)$.

Solution of a switched system

$$\begin{cases} \dot{x}(t) = f_{\sigma(t)}(x(t)), \ t \geq 0, \\ x(0) = x_0, \end{cases}$$

Denote the solution $\varphi(t; x_0, \sigma)$.

$\sigma(t) = i_0$, for $t_0 = 0 \leq t < t_1$.

$\bullet x_0$

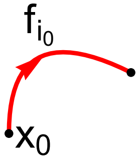
Solution of a switched system

$$\begin{cases} \dot{x}(t) = f_{\sigma(t)}(x(t)), \ \ t \geq 0, \\ x(0) = x_0, \end{cases}$$

Denote the solution $\varphi(t; x_0, \sigma)$.

$\sigma(t) = i_0$, for $t_0 = 0 \leq t < t_1$.

$$\begin{cases} \dot{x}(t) = f_{i_0}(x(t)), \ \ t_0 \leq t \leq t_1 \\ x(0) = x_0. \end{cases}$$



Solution of a switched system

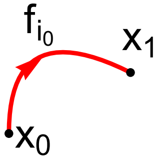
$$\begin{cases} \dot{x}(t) = f_{\sigma(t)}(x(t)), \ t \geq 0, \\ x(0) = x_0, \end{cases}$$

Denote the solution $\varphi(t; x_0, \sigma)$.

$\sigma(t) = i_0$, for $t_0 = 0 \leq t < t_1$.

$$\begin{cases} \dot{x}(t) = f_{i_0}(x(t)), \ t_0 \leq t \leq t_1 \\ x(0) = x_0. \end{cases}$$

$x_1 = \varphi(t_1; x_0, \sigma)$.



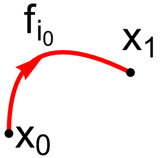
Solution of a switched system

$$\begin{cases} \dot{x}(t) = f_{\sigma(t)}(x(t)), \ t \geq 0, \\ x(0) = x_0, \end{cases}$$

Denote the solution $\varphi(t; x_0, \sigma)$.

$\sigma(t) = i_0$, for $t_0 = 0 \leq t < t_1$.

$$\begin{cases} \dot{x}(t) = f_{i_0}(x(t)), \ t_0 \leq t \leq t_1 \\ x(0) = x_0. \end{cases}$$



$x_1 = \varphi(t_1; x_0, \sigma)$.
$\sigma(t) = i_1$, for $t_1 \leq t < t_2$.

Solution of a switched system

$$\begin{cases} \dot{x}(t) = f_{\sigma(t)}(x(t)), \; t \geq 0, \\ x(0) = x_0, \end{cases}$$

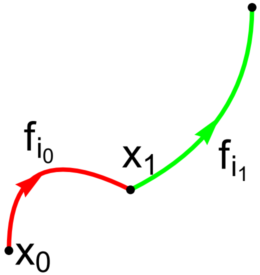
Denote the solution $\varphi(t; x_0, \sigma)$.

$\sigma(t) = i_0$, for $t_0 = 0 \leq t < t_1$.

$$\begin{cases} \dot{x}(t) = f_{i_0}(x(t)), \; t_0 \leq t \leq t_1 \\ x(0) = x_0. \end{cases}$$

$x_1 = \varphi(t_1; x_0, \sigma)$.
$\sigma(t) = i_1$, for $t_1 \leq t < t_2$.

$$\begin{cases} \dot{x}(t) = f_{i_1}(x(t)), \; t_1 \leq t \leq t_2 \\ x(t_1) = x_1. \end{cases}$$



Solution of a switched system

$$\begin{cases} \dot{x}(t) = f_{\sigma(t)}(x(t)), \ t \geq 0, \\ x(0) = x_0, \end{cases}$$

Denote the solution $\varphi(t; x_0, \sigma)$.

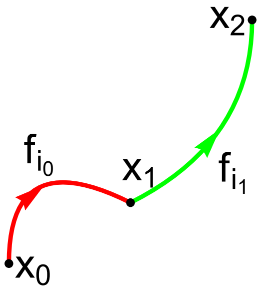
$\sigma(t) = i_0$, for $t_0 = 0 \leq t < t_1$.

$$\begin{cases} \dot{x}(t) = f_{i_0}(x(t)), \ t_0 \leq t \leq t_1 \\ x(0) = x_0. \end{cases}$$

$x_1 = \varphi(t_1; x_0, \sigma)$.
$\sigma(t) = i_1$, for $t_1 \leq t < t_2$.

$$\begin{cases} \dot{x}(t) = f_{i_1}(x(t)), \ t_1 \leq t \leq t_2 \\ x(t_1) = x_1. \end{cases}$$

$x_2 = \varphi(t_2; x_0, \sigma)$.



Example of second order switched system

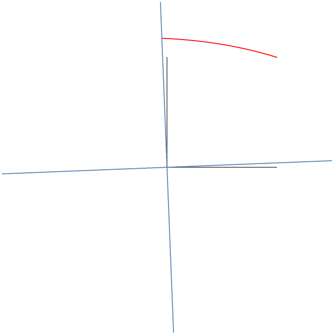
$$A_1 = \begin{pmatrix} .1 & -2 \\ .5 & .1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} .1 & -.5 \\ 2 & .1 \end{pmatrix}$$

Example of second order switched system

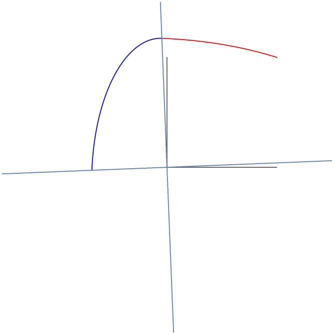
$$A_1 = \begin{pmatrix} .1 & -2 \\ .5 & .1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} .1 & -.5 \\ 2 & .1 \end{pmatrix}$$

- Eigenvalues of A_1: $.1 \pm i$
- Eigenvalues of A_2: $.1 \pm i$

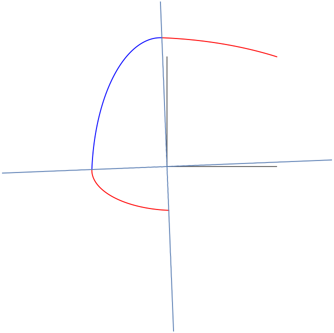
Example of second order switched system



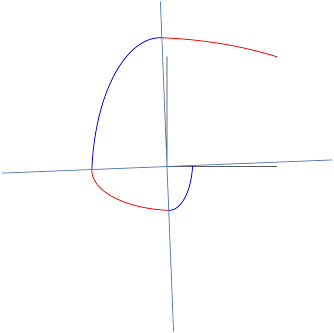
Example of second order switched system



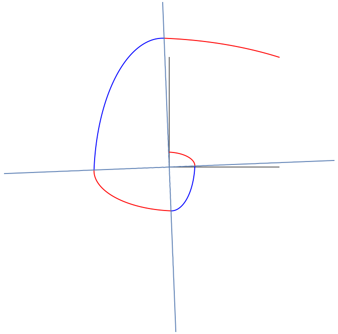
Example of second order switched system



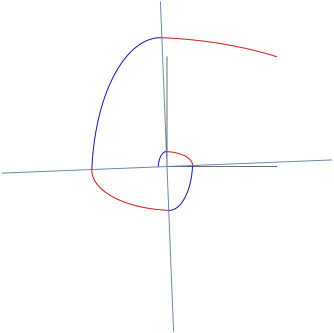
Example of second order switched system



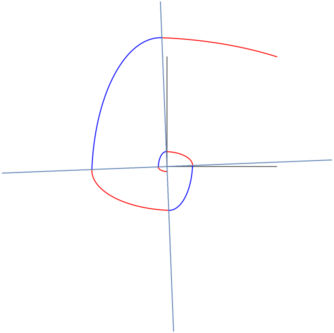
Example of second order switched system



Example of second order switched system



Example of second order switched system



Problems on switched systems

1. Reachability problem.

Problems on switched systems

1. Reachability problem.
2. Invariant set.

Problems on switched systems

1. Reachability problem.
2. Invariant set.
3. Stabilization problem.

Outline of this section

2. **Invariant sets for third order switched systems**
   - Preliminaries
   - Invariant octant for simplified case
   - Numerical example

Preliminaries

A third order switched linear system with three subsystems

$$\dot{x} = A_k x, \quad k = 1, 2, 3,$$

where A_1, A_2 and A_3 are 3×3 real matrices.

Preliminaries

A third order switched linear system with three subsystems

$$\dot{x} = A_k x, \quad k = 1, 2, 3,$$

where A_1, A_2 and A_3 are 3×3 real matrices.

1. Eigenvalues of A_k:

   $$\lambda_k, a_k \pm b_k i$$

   with $\lambda_k, a_k, b_k \in \mathbb{R}$ and $b_k \neq 0$, $k = 1, 2, 3$.

2. $$v_1, v_2, v_3$$

   are linearly independent, where v_k is the eigenvector of A_k associated to λ_k, $k = 1, 2, 3$.

Preliminaries

Invariant set

A set $S \subset \mathbb{R}^3$ is an **invariant set** for the switched system if for each $x_0 \in S$ there is a switching law σ such that

$$\varphi(t; x_0, \sigma) \in S, \qquad t \geq 0.$$

Preliminaries

Let $S \subset \mathbb{R}^3$ be a closed set, P be a non-singular matrix and σ be a feedback switching law.

$P(S)$ is invariant for the switched system

$$\dot{x} = A_{\sigma \circ P^{-1}} x,$$

if, and only if,

S is invariant for the switched system

$$\dot{y} = P^{-1} A_{\sigma} P y.$$

Preliminaries

Let $S \subset \mathbb{R}^3$ be a closed set, P be a non-singular matrix and σ be a feedback switching law.

$P(S)$ is invariant for the switched system

$$\dot{x} = A_{\sigma \circ P^{-1}} x,$$

if, and only if,

S is invariant for the switched system

$$\dot{y} = P^{-1} A_\sigma P y.$$

We denote $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

Preliminaries

Let $S \subset \mathbb{R}^3$ be a closed set, P be a non-singular matrix and σ be a feedback switching law.

$P(S)$ is invariant for the switched system

$$\dot{x} = A_{\sigma \circ P^{-1}} x,$$

if, and only if,

S is invariant for the switched system

$$\dot{y} = P^{-1} A_\sigma P y.$$

We denote $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$.

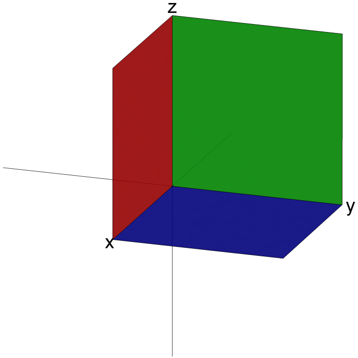
We can suppose $v_k = e_k$, $k = 1,2,3$.

Invariant octant for simplified case

Every octant of $\mathbb{R}^3$ is identified with three signs.
Denote $O(a, b, c)$ the octant with signs $a, b, c \in \{-1, +1\}$.

Invariant octant for simplified case

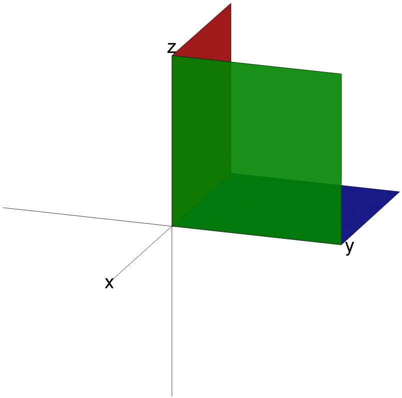
Every octant of $\mathbb{R}^3$ is identified with three signs.
Denote $O(a, b, c)$ the octant with signs $a, b, c \in \{-1, +1\}$.



Octant $O(+, +, +)$.

Invariant octant for simplified case

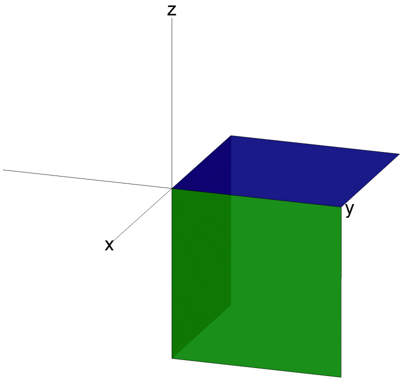
Every octant of $\mathbb{R}^3$ is identified with three signs.
Denote $O(a, b, c)$ the octant with signs $a, b, c \in \{-1, +1\}$.



Octant $O(-, +, +)$.

Invariant octant for simplified case

Every octant of $\mathbb{R}^3$ is identified with three signs.
Denote $O(a, b, c)$ the octant with signs $a, b, c \in \{-1, +1\}$.



Octant $O(-, +, -)$.

Invariant octant for simplified case

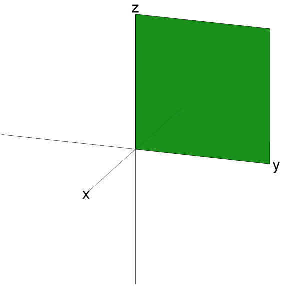
The faces of $O(a,b,c)$ are

$$O(0,b,c) = \{(0,x_2,x_3) : bx_2 > 0, cx_3 > 0\},$$
$$O(a,0,c) = \{(x_1,0,x_3) : ax_1 > 0, cx_3 > 0\},$$
$$O(a,b,0) = \{(x_1,x_2,0) : ax_1 > 0, bx_2 > 0\},$$
$$O(a,0,0) = \{(x_1,0,0) : ax_1 > 0\},$$
$$O(0,b,0) = \{(0,x_2,0) : bx_2 > 0\},$$
$$O(0,0,c) = \{(0,0,x_3) : cx_3 > 0\}.$$

Invariant octant for simplified case

The faces of $O(a, b, c)$ are

$$O(0, b, c) = \{(0, x_2, x_3) : bx_2 > 0, cx_3 > 0\},$$
$$O(a, 0, c) = \{(x_1, 0, x_3) : ax_1 > 0, cx_3 > 0\},$$
$$O(a, b, 0) = \{(x_1, x_2, 0) : ax_1 > 0, bx_2 > 0\},$$
$$O(a, 0, 0) = \{(x_1, 0, 0) : ax_1 > 0\},$$
$$O(0, b, 0) = \{(0, x_2, 0) : bx_2 > 0\},$$
$$O(0, 0, c) = \{(0, 0, x_3) : cx_3 > 0\}.$$

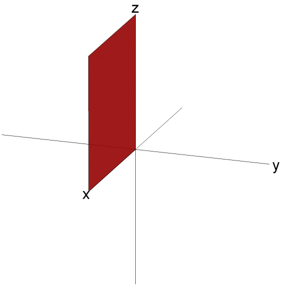


Octant $O(0, +, +)$.

Invariant octant for simplified case

The faces of $O(a, b, c)$ are

$$O(0, b, c) = \{(0, x_2, x_3) : bx_2 > 0, cx_3 > 0\},$$
$$O(a, 0, c) = \{(x_1, 0, x_3) : ax_1 > 0, cx_3 > 0\},$$
$$O(a, b, 0) = \{(x_1, x_2, 0) : ax_1 > 0, bx_2 > 0\},$$
$$O(a, 0, 0) = \{(x_1, 0, 0) : ax_1 > 0\},$$
$$O(0, b, 0) = \{(0, x_2, 0) : bx_2 > 0\},$$
$$O(0, 0, c) = \{(0, 0, x_3) : cx_3 > 0\}.$$

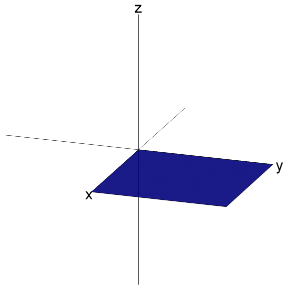


Octant $O(+, 0, +)$.

Invariant octant for simplified case

The faces of $O(a, b, c)$ are

$$O(0, b, c) = \{(0, x_2, x_3) : bx_2 > 0, cx_3 > 0\},$$
$$O(a, 0, c) = \{(x_1, 0, x_3) : ax_1 > 0, cx_3 > 0\},$$
$$O(a, b, 0) = \{(x_1, x_2, 0) : ax_1 > 0, bx_2 > 0\},$$
$$O(a, 0, 0) = \{(x_1, 0, 0) : ax_1 > 0\},$$
$$O(0, b, 0) = \{(0, x_2, 0) : bx_2 > 0\},$$
$$O(0, 0, c) = \{(0, 0, x_3) : cx_3 > 0\}.$$



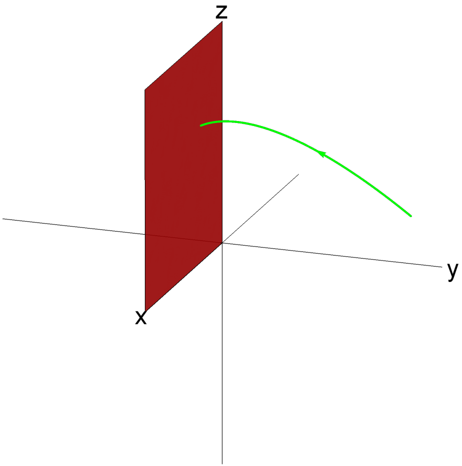
Octant $O(+, +, 0)$.

Invariant octant for simplified case

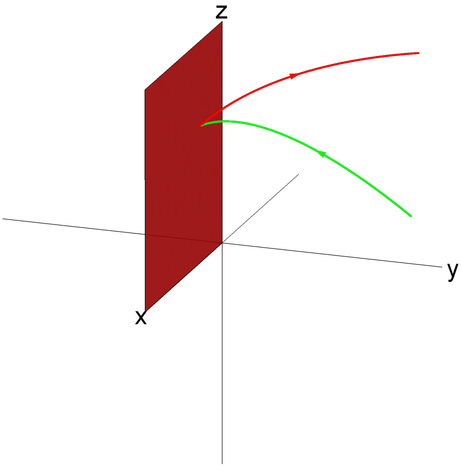
Feedback switching laws σ_1 and σ_2

$$\sigma_1(x) = \begin{cases} 3 & \text{if } x \in O(0,b,c) \\ 1 & \text{if } x \in O(a,0,c) \\ 2 & \text{if } x \in O(a,b,0) \\ 3 & \text{if } x \in O(0,0,c) \\ 2 & \text{if } x \in O(0,b,0) \\ 1 & \text{if } x \in O(a,0,0) \end{cases} \qquad \sigma_2(x) = \begin{cases} 2 & \text{if } x \in O(0,b,c) \\ 3 & \text{if } x \in O(a,0,c) \\ 1 & \text{if } x \in O(a,b,0) \\ 3 & \text{if } x \in O(0,0,c) \\ 2 & \text{if } x \in O(0,b,0) \\ 1 & \text{if } x \in O(a,0,0) \end{cases}$$

Invariant octant for simplified case



Invariant octant for simplified case



Invariant octant for simplified case

Invariant octant for σ_1

1. $O(a, b, c)$ is invariant for the switched system with face switching law σ_1 if, and only if,
2. $\text{sign}(a_{12}^3) = \text{sign}(ab)$, $\text{sign}(a_{23}^1) = \text{sign}(bc)$ and $\text{sign}(a_{31}^2) = \text{sign}(ac)$.

Invariant octant for σ_2

1. $O(a, b, c)$ is invariant for the switched system with face switching law σ_2 if, and only if,
2. $\text{sign}(a_{12}^3) = -\text{sign}(ab)$, $\text{sign}(a_{23}^1) = -\text{sign}(bc)$ and $\text{sign}(a_{31}^2) = -\text{sign}(ac)$.

Invariant octant for simplified case

a_{23}^1	a_{31}^2	a_{12}^3		
$+$	$+$	$+$	σ_1	$O(+,+,+), O(-,-,-)$
$+$	$+$	$-$	σ_2	$O(+,+,-), O(-,-,+)$
$+$	$-$	$+$	σ_2	$O(+,-,+), O(-,+,-)$
$+$	$-$	$-$	σ_1	$O(+,-,-), O(-,+,+)$
$-$	$+$	$+$	σ_2	$O(+,-,-), O(-,+,+)$
$-$	$+$	$-$	σ_1	$O(+,-,+), O(-,+,-)$
$-$	$-$	$+$	σ_1	$O(+,+,-), O(-,-,+)$
$-$	$-$	$-$	σ_2	$O(+,+,+), O(-,-,-)$

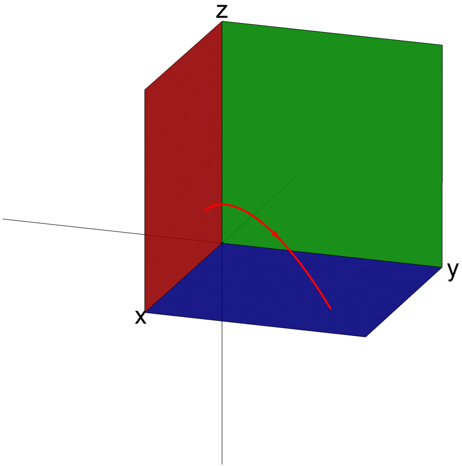
Numerical example

$$A_1 = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & -2 & 1 \end{pmatrix}, \; A_2 = \begin{pmatrix} -1 & 0 & -10 \\ -3 & 0 & -5 \\ 1 & 0 & 5 \end{pmatrix}, \; A_3 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}.$$

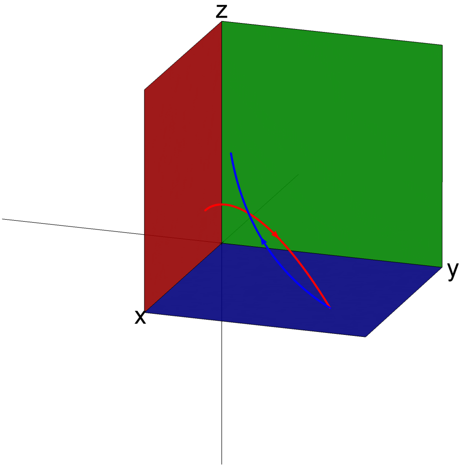
Numerical example

$$A_1 = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 0 & -2 & 1 \end{pmatrix}, \; A_2 = \begin{pmatrix} -1 & 0 & -10 \\ -3 & 0 & -5 \\ 1 & 0 & 5 \end{pmatrix}, \; A_3 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}.$$

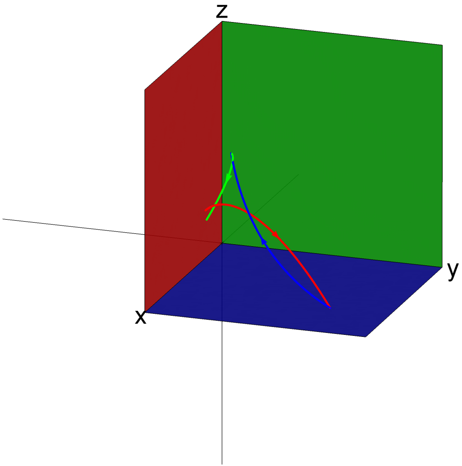
Numerical example



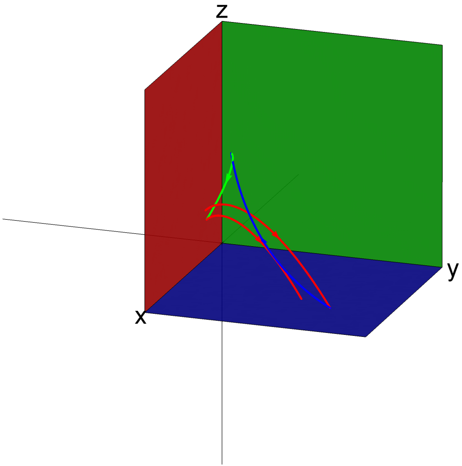
Numerical example



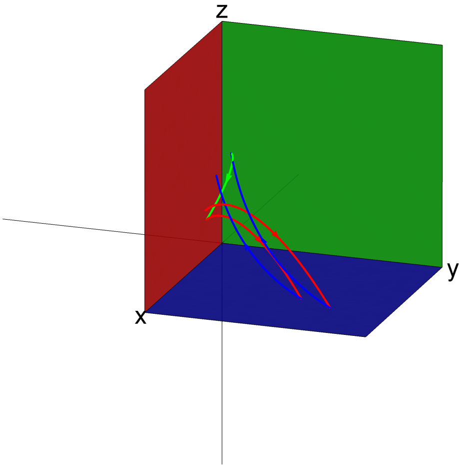
Numerical example



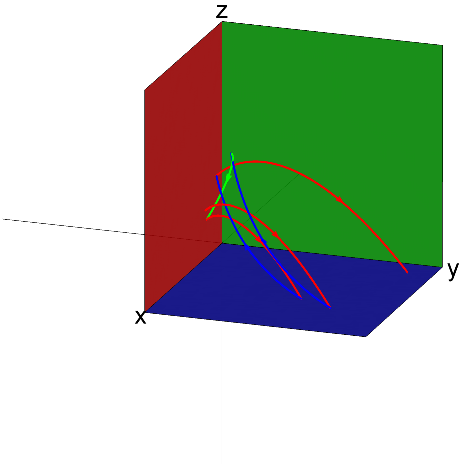
Numerical example



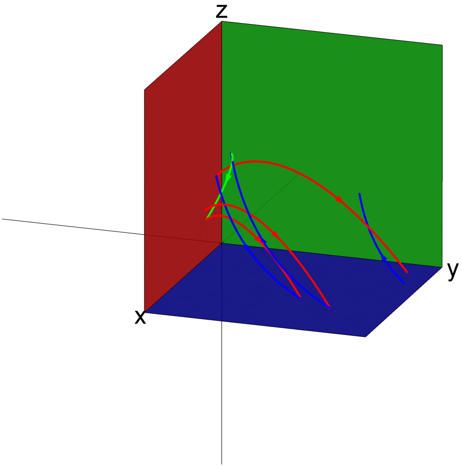
Numerical example



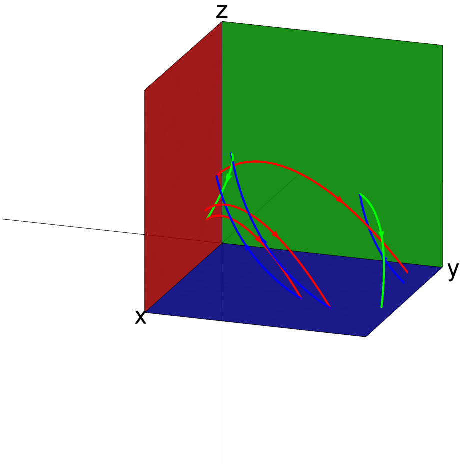
Numerical example



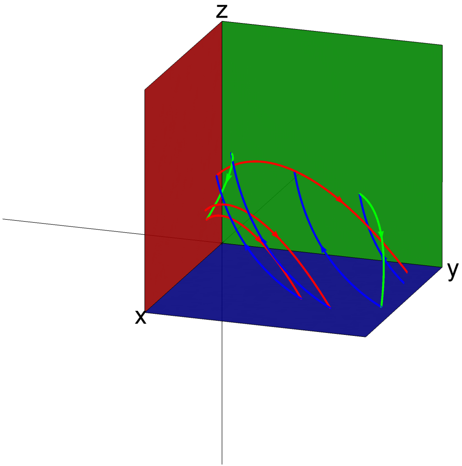
Numerical example



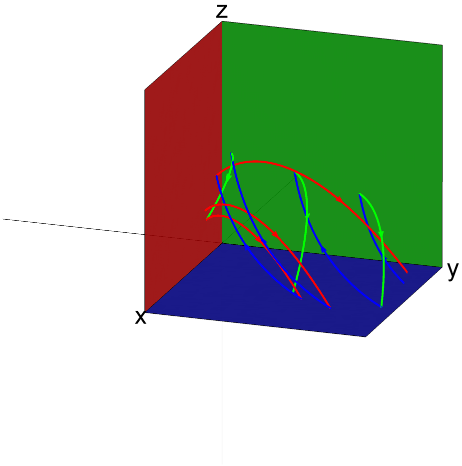
Numerical example



Numerical example



Numerical example



Outline of this section

3. **A stabilization condition for third order switched systems**
   - Preliminaries
   - Main result
   - Numerical example

Preliminaries

A third order switched linear system with three subsystems

$$\dot{x} = A_k x, \quad k = 1, 2, 3,$$

where A_1, A_2 and A_3 are 3×3 real matrices.

1. Eigenvalues of A_k:

$$\lambda_k, a_k \pm b_k i$$

   with $\lambda_k, a_k, b_k \in \mathbb{R}$ and $b_k \neq 0$, $k = 1, 2, 3$.
2. e_k is the eigenvector of A_k associated to λ_k, $k = 1, 2, 3$, where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

Preliminaries

The positive octant $O = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \geq 0\}$ is invariant for the switching law

$$\sigma(x) = \begin{cases} 1 & \text{if } \; x \in C_1 \cup E_1, \\ 2 & \text{if } \; x \in C_2 \cup E_2, \\ 3 & \text{if } \; x \in C_3 \cup E_3, \\ \text{unchanged} & \text{if } \; x \in \text{int}(O). \end{cases}$$

where

- $C_1 = \{(x_1, 0, x_3) \in \mathbb{R}^3 : x_1 > 0, x_3 > 0\}$,
- $C_2 = \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1 > 0, x_2 > 0\}$,
- $C_3 = \{(0, x_2, x_3) \in \mathbb{R}^3 : x_2 > 0, x_3 > 0\}$,
- $E_1 = \{(x_1, 0, 0) \in \mathbb{R}^3 : x_1 > 0\}$,
- $E_2 = \{(0, x_2, 0) \in \mathbb{R}^3 : x_2 > 0\}$,
- $E_3 = \{(0, 0, x_3) \in \mathbb{R}^3 : x_3 > 0\}$.

Preliminaries

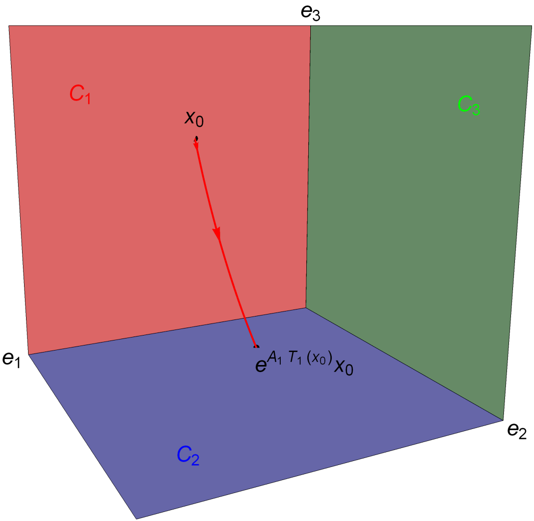
We consider the following functions

$$T_1 : C_1 \to \mathbb{R}, \quad T_2 : C_2 \to \mathbb{R}, \quad T_3 : C_3 \to \mathbb{R},$$

defined as

$$\begin{aligned}
T_1(x_0) &= \inf\{T > 0 : e^{A_1 T} x_0 \in F_2 \cup F_3\}, \text{ for } x_0 \in C_1,\\
T_2(x_0) &= \inf\{T > 0 : e^{A_2 T} x_0 \in F_1 \cup F_3\}, \text{ for } x_0 \in C_2,\\
T_3(x_0) &= \inf\{T > 0 : e^{A_3 T} x_0 \in F_1 \cup F_2\}, \text{ for } x_0 \in C_3.
\end{aligned}$$

Preliminaries



Preliminaries

Proposition

If $x_0, y_0 \in C_k$ such that $x_0 = \mu y_0$ with $\mu > 0$ then

$$T_k(x_0) = T_k(y_0).$$

Preliminaries

1. $e^{A_1 T_1(x_0)} x_0 \in F_2$, for $x_0 \in C_1$,
2. $e^{A_2 T_2(x_0)} x_0 \in F_3$, for $x_0 \in C_2$,
3. $e^{A_3 T_3(x_0)} x_0 \in F_1$, for $x_0 \in C_3$.

Preliminaries

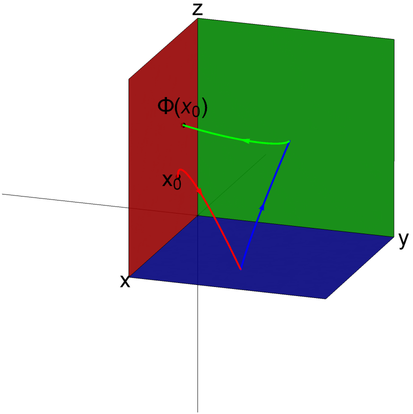
1. $e^{A_1 T_1(x_0)} x_0 \in F_2$, for $x_0 \in C_1$,
2. $e^{A_2 T_2(x_0)} x_0 \in F_3$, for $x_0 \in C_2$,
3. $e^{A_3 T_3(x_0)} x_0 \in F_1$, for $x_0 \in C_3$.

Proposition

It turns out that $T_k(x_0) = T_k(y_0)$ for all $x_0, y_0 \in C_k$, for $k = 1, 2, 3$.

Preliminaries

$$\Phi(x_0) = e^{A_3 T_3} e^{A_2 T_2} e^{A_1 T_1} x_0, \quad x_0 \in F_1.$$



Preliminaries

$$M = \Phi_{|\mathcal{B}},$$

where $\mathcal{B} = \{e_1, e_3\}$.

Preliminaries

$$M = \Phi_{|\mathcal{B}},$$

where $\mathcal{B} = \{e_1, e_3\}$.

Frobenius Theorem

An irreducible non negative matrix M has a positive simple eigenvalue r, there is an eigenvector of r with positive coordinates, and the other eigenvalues have modulus less or equal than r.

Main result

Theorem

If the previous eigenvalue r of M is less than 1, then the switched system is stabilizable with the switching law σ.

Numerical example

Consider a third order switched system with matrices

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2.5 \\ 0 & -20 & 1 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1 & 0 & -12 \\ 0 & 1 & 0 \\ 0.5 & 0 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & 1 & 0 \\ -12 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Numerical example

Consider a third order switched system with matrices

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2.5 \\ 0 & -20 & 1 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1 & 0 & -12 \\ 0 & 1 & 0 \\ 0.5 & 0 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & 1 & 0 \\ -12 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- Eigenvalues of A_1: $1, 1 \pm 5\sqrt{2}i$.
- Eigenvalues of A_2: $1, 1 \pm \sqrt{5}i$.
- Eigenvalues of A_3: $1, 1 \pm 2\sqrt{3}i$.

Numerical example

Consider a third order switched system with matrices

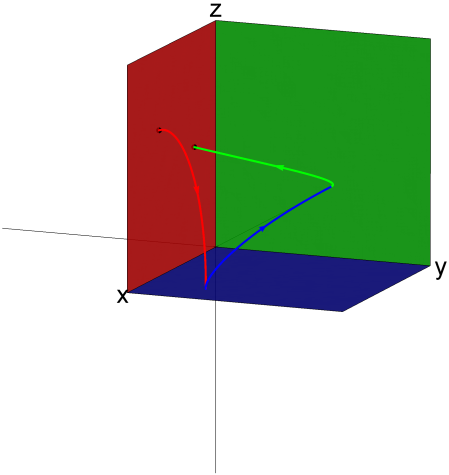
$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2.5 \\ 0 & -20 & 1 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 1 & 0 & -12 \\ 0 & 1 & 0 \\ 0.5 & 0 & 1 \end{pmatrix},$$

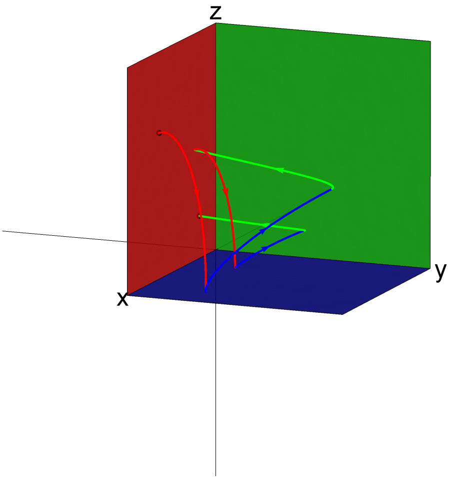
$$A_3 = \begin{pmatrix} 1 & 1 & 0 \\ -12 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- Eigenvalues of A_1: $1, 1 \pm 5\sqrt{2}i$.
- Eigenvalues of A_2: $1, 1 \pm \sqrt{5}i$.
- Eigenvalues of A_3: $1, 1 \pm 2\sqrt{3}i$.
- $T_1 = 0.222144$, $T_2 = 0.641275$ and $T_3 = 0.45345$.
- $r = 0.538627$.
- Since $r < 1$, the result assures that the system is stabilizable.

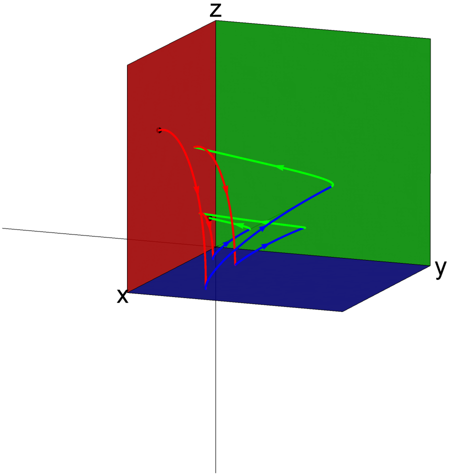
Numerical example



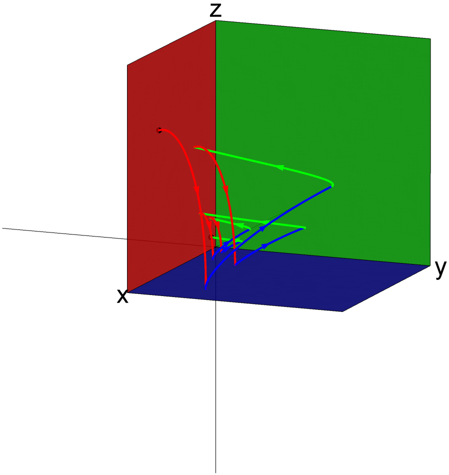
Numerical example



Numerical example



Numerical example



Outline of this section

4. **A new method for stabilization**
   - Preliminaries
   - Main result
   - Handling Zeno-behavior
   - Numerical example

Preliminaries

Given a switched linear system with subsystems

$$\dot{x} = A_i x, \quad i = 1, \dots, N$$

where A_i is a $n \times n$ real matrix for $i = 1, \dots, N$.

Preliminaries

Given a switched linear system with subsystems

$$\dot{x} = A_i x, \quad i = 1, \ldots, N$$

where A_i is a $n \times n$ real matrix for $i = 1, \ldots, N$.

Given a switching law σ. The switching times of σ

$$0 = t_0 < t_1 < \cdots < t_k < \cdots .$$

Then

$$\sigma(t) = i_k, \quad \text{for } t_k \leq t < t_{k+1},$$

for all $k \geq 0$, where $i_k \in \{1, \ldots, N\}$.

Preliminaries

Given a switched linear system with subsystems

$$\dot{x} = A_i x, \quad i = 1, \ldots, N$$

where A_i is a $n \times n$ real matrix for $i = 1, \ldots, N$.

Given a switching law σ. The switching times of σ

$$0 = t_0 < t_1 < \cdots < t_k < \cdots .$$

Then

$$\sigma(t) = i_k, \quad \text{for } t_k \leq t < t_{k+1},$$

for all $k \geq 0$, where $i_k \in \{1, \ldots, N\}$.

The solution is

$$\varphi(t; x_0, \sigma) = e^{A_{i_k}(t - t_k)} e^{A_{i_{k-1}}(t_k - t_{k-1})} \ldots e^{A_{i_1}(t_2 - t_1)} e^{A_{i_0} t_1} x_0,$$

for $t_k \leq t \leq t_{k+1}$, $k \geq 0$.

Preliminaries

Given a switched linear system with subsystems

$$\dot{x} = A_i x, \quad i = 1, \dots, N$$

where A_i is a $n \times n$ real matrix for $i = 1, \dots, N$.

Given a switching law σ. The switching times of σ

$$0 = t_0 < t_1 < \cdots < t_k < \cdots .$$

Then

$$\sigma(t) = i_k, \quad \text{for } t_k \leq t < t_{k+1},$$

for all $k \geq 0$, where $i_k \in \{1, \dots, N\}$.

The solution is

$$\varphi(t; x_0, \sigma) = e^{A_{i_k}(t-t_k)} e^{A_{i_{k-1}} h_k} \cdots e^{A_{i_1} h_2} e^{A_{i_0} h_1} x_0,$$

for $t_k \leq t \leq t_{k+1}$, $k \geq 0$.

Preliminaries

We are looking for conditions for the property

$$\lim_{t\to+\infty} \varphi(t; x_0, \sigma) = 0.$$

Preliminaries

We are looking for conditions for the property

$$\lim_{t\to+\infty} \varphi(t; x_0, \sigma) = 0.$$

Why not minimize the function f_k?

$$f_k(t_1, t_2, \dots, t_N, t_{N+1}, \dots, t_{kN}) = \|e^{A_N t_{kN}} \cdots e^{A_N t_N} \cdots e^{A_2 t_2} e^{A_1 t_1} x_0\|^2$$

for $t_1, \dots, t_{kN} \geq 0$.

Main result

Theorem

Let A_1, A_2, ..., A_N be real $n \times n$ matrices.

1. If the switched system is stabilizable then for any μ, with $0 < \mu < 1$, there exists $k \in \mathbb{N}$ such that for each $x_0 \in \mathbb{R}^n \setminus \{0\}$ the function f_k has a minimum at $h = (h_1, h_2, \ldots, h_{kN})$ with $h \neq 0$ such that

$$f_k(h) \leq \mu^2 \|x_0\|^2$$

Main result

Theorem

Let A_1, A_2, ..., A_N be real $n \times n$ matrices.

1. If the switched system is stabilizable then for any μ, with $0 < \mu < 1$, there exists $k \in \mathbb{N}$ such that for each $x_0 \in \mathbb{R}^n \setminus \{0\}$ the function f_k has a minimum at $h = (h_1, h_2, \dots, h_{kN})$ with $h \neq 0$ such that

$$f_k(h) \leq \mu^2 \|x_0\|^2$$

2. If there exist μ, with $0 < \mu < 1$, and $k \in \mathbb{N}$ such that for any $x_0 \in \mathbb{R}^n \setminus \{0\}$ the function f_k has a minimum at $h = (h_1, h_2, \dots, h_{kN}) \neq 0$ with $h_i \geq 0$ such that

$$f_k(h) \leq \mu^2 \|x_0\|^2$$

then the switched system is stabilizable.

Main result

How can the switching law be defined?

Main result

How can the switching law be defined?

Let $x_0 \in \mathbb{R}^n$ be a convergent state.

Main result

How can the switching law be defined?

Let $x_0 \in \mathbb{R}^n$ be a convergent state.

- For $0 < \mu < 1$ fixed, there is $k \in \mathbb{N}$ such that for x_0 there is $h^0 = (h_1^0, h_2^0, \ldots, h_{kN}^0)$ such that

$$f_k(h^0) \leq \mu^2 \|x_0\|^2.$$

Main result

How can the switching law be defined?

Let $x_0 \in \mathbb{R}^n$ be a convergent state.

- For $0 < \mu < 1$ fixed, there is $k \in \mathbb{N}$ such that for x_0 there is $h^0 = (h_1^0, h_2^0, \ldots, h_{kN}^0)$ such that

$$f_k(h^0) \leq \mu^2 \|x_0\|^2.$$

- The switching law for time $0 = t_0 \leq t \leq t_1 = \sum_{i=1}^{kN} h_i^0$ is

$$\sigma(t) = \begin{cases} 1 & \text{if} \quad t \in [0, h_1^0) \\ 2 & \text{if} \quad t \in [h_1^0, h_1^0 + h_2^0) \\ \vdots & \\ N & \text{if} \quad t \in \left[\sum_{i=1}^{kN-1} h_i^0, \sum_{i=1}^{kN} h_i^0\right) \end{cases}$$

Main result

How can the switching law be defined?

Let $x_0 \in \mathbb{R}^n$ be a convergent state.

- For $0 < \mu < 1$ fixed, there is $k \in \mathbb{N}$ such that for x_0 there is $h^0 = (h_1^0, h_2^0, \ldots, h_{kN}^0)$ such that

$$f_k(h^0) \leq \mu^2 \|x_0\|^2.$$

- The switching law for time $0 = t_0 \leq t \leq t_1 = \sum_{i=1}^{kN} h_i^0$ is

$$\sigma(t) = \begin{cases} 1 & \text{if} \quad t \in [0, h_1^0) \\ 2 & \text{if} \quad t \in [h_1^0, h_1^0 + h_2^0) \\ \vdots & \\ N & \text{if} \quad t \in \left[\sum_{i=1}^{kN-1} h_i^0, \sum_{i=1}^{kN} h_i^0\right) \end{cases}$$

- The state $x_1 = \varphi(t_1; x_0, \sigma)$ verifies

$$\|x_1\|^2 \leq \mu^2 \|x_0\|^2.$$

Main result

- Again, for x_1 there is $h^1 = (h_1^1, h_2^1, \ldots, h_{kN}^1)$ such that

$$f_k(h^1) \leq \mu^2 \|x_1\|^2.$$

Main result

- Again, for x_1 there is $h^1 = (h_1^1, h_2^1, \ldots, h_{kN}^1)$ such that
$$f_k(h^1) \leq \mu^2 \|x_1\|^2.$$
- The switching law for time $t_1 \leq t \leq t_2 = t_1 + \sum_{i=1}^{kN} h_i^1$ is

$$\sigma(t) = \begin{cases} 1 & \text{if} \quad t \in [t_1 + 0, t_1 + h_1^1) \\ 2 & \text{if} \quad t \in [t_1 + h_1^1, t_1 + h_1^1 + h_2^1) \\ \vdots & \\ N & \text{if} \quad t \in \left[t_1 + \sum_{i=1}^{kN-1} h_i^1, t_1 + \sum_{i=1}^{kN} h_i^1\right) \end{cases}$$

Main result

- Again, for x_1 there is $h^1 = (h^1_1, h^1_2, \dots, h^1_{kN})$ such that

$$f_k(h^1) \leq \mu^2 \|x_1\|^2.$$

- The switching law for time $t_1 \leq t \leq t_2 = t_1 + \sum_{i=1}^{kN} h^1_i$ is

$$\sigma(t) = \begin{cases} 1 & \text{if} \quad t \in [t_1 + 0, t_1 + h^1_1) \\ 2 & \text{if} \quad t \in [t_1 + h^1_1, t_1 + h^1_1 + h^1_2) \\ \vdots & \\ N & \text{if} \quad t \in \left[t_1 + \sum_{i=1}^{kN-1} h^1_i, t_1 + \sum_{i=1}^{kN} h^1_i\right) \end{cases}$$

- The state $x_2 = \varphi(t_2; x_0, \sigma)$ verifies

$$\|x_2\|^2 \leq \mu^2 \|x_1\|^2 \leq \mu^4 \|x_0\|^2.$$

Handling Zeno-behavior

Therefore, with the defined switching law σ the solution verifies

$$\lim_{t \to +\infty} \varphi(t; x_0, \sigma) = 0.$$

Handling Zeno-behavior

Therefore, with the defined switching law σ the solution verifies

$$\lim_{t \to ?} \varphi(t; x_0, \sigma) = 0.$$

The previous result does not assure that

$$\lim_{m} \left(\sum_{i=1}^{kN} h_i^0 + \cdots + \sum_{i=1}^{kN} h_i^m \right) = +\infty.$$

Handling Zeno-behavior

Theorem

Let A_1, A_2, ..., A_N be real $n \times n$ matrices.

1. If the switched system is stabilizable then for any μ, with $0 < \mu < 1$, there exist $k \in \mathbb{N}$ and $\tau > 0$ such that for each $x_0 \in \mathbb{R}^n \setminus \{0\}$ the function f_k has a minimum at $h = (h_1, h_2, \ldots, h_{kN})$ with $h \neq 0$ such that

$$f_k(h) \leq \mu^2 \|x_0\|^2 \qquad \text{and} \qquad h_1 + h_2 + \cdots + h_{kN} \geq \tau.$$

2. If there exist μ, with $0 < \mu < 1$, $\tau > 0$ and $k \in \mathbb{N}$ such that for any $x_0 \in \mathbb{R}^n \setminus \{0\}$ the function f_k has a minimum at $h = (h_1, h_2, \ldots, h_{kN}) \neq 0$ with $h_i \geq 0$ such that

$$f_k(h) \leq \mu^2 \|x_0\|^2 \qquad \text{and} \qquad h_1 + h_2 + \cdots + h_{kN} \geq \tau$$

then the switched system is stabilizable.

Handling Zeno-behavior

Theorem

Let A_1, A_2, ..., A_N be real $n \times n$ matrices.

1. If the switched system is stabilizable then for any μ, with $0 < \mu < 1$, there exist $k \in \mathbb{N}$ and $\tau > 0$ such that for each $x_0 \in \mathbb{R}^n \setminus \{0\}$ the function f_k has a minimum at $h = (h_1, h_2, \dots, h_{kN})$ with $h \neq 0$ such that
$$f_k(h) \leq \mu^2 \|x_0\|^2 \qquad \text{and} \qquad h_1 + h_2 + \cdots + h_{kN} \geq \tau.$$
2. If there exist μ, with $0 < \mu < 1$, $\tau > 0$ and $k \in \mathbb{N}$ such that for any $x_0 \in \mathbb{R}^n \setminus \{0\}$ the function f_k has a minimum at $h = (h_1, h_2, \dots, h_{kN}) \neq 0$ with $h_i \geq 0$ such that
$$f_k(h) \leq \mu^2 \|x_0\|^2 \qquad \text{and} \qquad h_1 + h_2 + \cdots + h_{kN} \geq \tau$$
then the switched system is stabilizable.

Numerical example

$$A_1 = \begin{pmatrix} -12 & 12 & -1 \\ -13 & 25 & 11 \\ 28 & -39 & -10 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 4 & 18 & 18 \\ 3 & 19 & 18 \\ -6 & -21 & -20 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 2 & -1 & 0 \\ 11 & 10 & 10 \\ -11 & -9 & -9 \end{pmatrix}.$$

Numerical example

$$A_1 = \begin{pmatrix} -12 & 12 & -1 \\ -13 & 25 & 11 \\ 28 & -39 & -10 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 4 & 18 & 18 \\ 3 & 19 & 18 \\ -6 & -21 & -20 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 2 & -1 & 0 \\ 11 & 10 & 10 \\ -11 & -9 & -9 \end{pmatrix}.$$

Eigenvalues of each matrix

$$\{1+\sqrt{6}i, 1-\sqrt{6}i, 1\}, \{1+3\sqrt{5}i, 1-3\sqrt{5}i, 1\}, \text{ and } \{1+\sqrt{10}i, 1-\sqrt{10}i, 1\}.$$

Numerical example

$$A_1 = \begin{pmatrix} -12 & 12 & -1 \\ -13 & 25 & 11 \\ 28 & -39 & -10 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 4 & 18 & 18 \\ 3 & 19 & 18 \\ -6 & -21 & -20 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 2 & -1 & 0 \\ 11 & 10 & 10 \\ -11 & -9 & -9 \end{pmatrix}.$$

Eigenvalues of each matrix

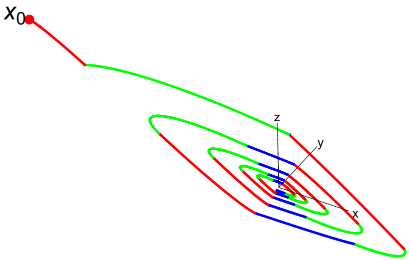
$$\{1+\sqrt{6}i, 1-\sqrt{6}i, 1\}, \{1+3\sqrt{5}i, 1-3\sqrt{5}i, 1\}, \text{ and } \{1+\sqrt{10}i, 1-\sqrt{10}i, 1\}.$$

Initial condition $x_0 = (-2, -2, 3)$.

$$h^0 = (h_1^0, h_2^0, h_3^0) = (0.05894, 0.142578, 2.140832 \cdot 10^{-12}).$$

$x_1 = e^{A_3 h_3^0} e^{A_2 h_2^0} e^{A_1 h_1^0} x_0$.

Numerical example



Estabilización de sistemas conmutados

Juan Bosco García Gutiérrez

V Jornadas Doctorales del Programa de Matemáticas

Universidad de Cádiz

Martes, 19 de noviembre de 2019