Algunos problemas no locales en EDP

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Local models

There are several biological and physical phenomena that can be modeled by PDEs

$$u_t(x,t) - Lu(x,t) = f(x,t,u(x,t))$$

$x \in \Omega$, bounded domain of $\mathbb{R}^N$, $t > 0$, L a linear second order elliptic operator.

In this problem, the relation between the unknown u and its derivatives are local in space.

There are, however, where a global spatial coupling is present in the phenomena and has to be incorporated in the model.

Some examples....

Liouville (1837) published a study of the equation

$$u_t = u_{xx} - b^2 x \int_0^1 x u_t dx$$

in connection with models in thermo-mechanics.

A turbulance model proposed by Burgers (1939)

$$\begin{cases} u_t = P - \dfrac{1}{R}u - \displaystyle\int_0^1 v^2 dx \\ v_t + 2v_x v = \dfrac{1}{R}v_{xx} + uv \end{cases}$$

Here u denotes the velocity in a channel due to some applied force P, while v stands for the turbulent perturbation of the motion and R is the Reynolds number associated with the viscosity of the fluid.

Burgers equation with non-local term

$$u_t = u_{xx} + \varepsilon u_x u + \frac{1}{2}(a\overline{u} + b)u$$

where $\overline{u}$ is a power of the L^p norm of u, that is,

$$\overline{u} = \left(\int_\Omega u^p\right)^q.$$

Non-Local models: examples

Plasma physics

$$u_t = \Delta u + \alpha \frac{e^{-u}}{\left(\int_\Omega e^{-u} dx \right)^p}$$

(when $p = 1$ Poisson-Boltzmann equation)

Non-local models: Kirchhoff equation

The Kirchhoff equation

$$u_t - M(x, \|u\|^2)\Delta u = f(x, u),$$

where

$$\|u\|^2 = \int_\Omega |\nabla u|^2.$$

(Alves, Correa, Ma (2005), G. Figueiredo....)

Non-Local models: examples

The distribution $u(x)$ of the phytoplankton species at water depth x is modeled by the following reduced model

$$-[du' - g(x)u]' = \left[f\left(e^{-k_0 x - \int_0^x u(\eta)d\eta}\right) - m\right] u$$

where

$$f(s) = \frac{s}{\delta_1 + s}.$$

Non-Local models: examples

The interaction of the grass density G and sediment height S:

$$\begin{cases} G_t - G_{xx} = G(F(S) - G) \\ S_t - S_{xx} = \phi(-L(G)S + 1) + \underbrace{\lambda S \int_{-\infty}^{\infty} P(x')G(x - x')dx'}_{\text{nonlocal deposition/erosion}}, \end{cases}$$

where $x \in \mathbb{R}$, $t > 0$,

$$F(s) = \frac{s - e_1}{s + p_1}, \quad L(s) = \frac{\delta s + e_3}{s + e_3},$$

and P is a Mexican-hat kernel

$$P(x) \simeq exp(-x^2/2\sigma_1^2) - exp(-x^2/2\sigma_2^2), \quad \sigma_1 < \sigma_2.$$

Furter and Grinfeld (1989) incorporated non-local effects in some population dynamics models:

"In ecological context, there is no real justification for assuming that the interactions are local. There are many (hypothetical) examples where such an assumption is clearly untenable, such as: (1) a population in which individua compete for a shared rapidly equilibrated (e.g. by convection) resource; (2) a population in which individua communicate either visually or by chemical means."

And they proposed, for example,

$$u_t - \Delta u = u \left(\lambda - a \int_\Omega u \right),$$

or even in a non-local diffusion:

$$u_t - A \left(\int_\Omega u \right) \Delta u = u(\lambda - au),$$

where A is a positive and regular function, see also Chipot (1996), Chipot& Lovat (1997), Chipot&Correa (2009), etc.....

Population dynamics and non-local terms: a more realistic model

We fix our attention in the reaction term. Specifically, in the carrying capacity.

The carrying capacity of the habitat is the level of population that can no longer be supported by the environment.

We present two different non-local models, equations, to consider more realistic models.

Population dynamics and non-local terms: a more realistic model

Chipot, 2006, proposed that the crowding effect depends also on the value of the population around of x, that is, the crowding effect depends on the value of u in a neighborhood of x, $B_r(x)$, the centered ball at x of radius $r > 0$. So, we consider the equation

$$\begin{cases} u_t - \Delta u = u\left(\lambda - \int_{\Omega\cap B_r(x)} b(y)u^p(y)dy\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

We can write

$$\int_{\Omega\cap B_r(x)} b(y)u^p(y)dy = \int_\Omega K(x,y)u^p(y)dy,$$

for some $K \in L^\infty(\Omega\times\Omega)$.

Population dynamics and non-local terms: a more realistic model

Britton,1989, consider that the species could exceed the carrying capacity in some part of the habitat and in others not. In this case, the species should consume resources in this last parts faster that they disappear. To model this situation, proposed to include a term as

$$\int_{\mathbb{R}^N} K(x,y)u^p(y)dy$$

in order to measure the growth rate at a point as a average of the population near the point. It is a natural choice to take

$$K(x,y) = g(|x-y|) = ce^{-\beta|x-y|}, \quad c, \beta > 0,$$

to model the fact that individuals in a population represent greater competition to a given individual the closer they are to that individual.

Population dynamics and non-local terms: a more realistic model

Or even in the diffusion

$$u_t - div\left(A\left(\int_{\Omega\cap B_r(x)} b(y)u^p(y)dy\right)\nabla u\right) = f(x,u)$$

Ovono, Rougirel (2010), Alves, Chipot, Correa (2016)

Population dynamics and non-local terms: birth-jump processes

Hillen et al. (2015) proposed the following general model

$$-d\Delta u = \underbrace{\int_\Omega K(x,y,u(x,t))\alpha(u(y,t))u(y,t)dy - \alpha(u(x,t))u(x,t)}_{\text{position-jump process}}$$

$$+ \underbrace{\int_\Omega S(x,y,u(x,t))\beta(u(y,t))u(y,t)dy}_{\text{birth-jump process}} - \underbrace{\delta(u(x,t))u(x,t)}_{\text{death}}$$

Population dynamics and non-local terms: birth-jump processes

1. The first two terms describe a nonlinear position-jump process, where $\alpha(u)$ is the rate for an individual to leave location x. The kernel K is a redistribution kernel representing the probability density of an individual to jump from y to x, conditioned on the local occupancy at x given by $u(x,t)$.
2. The third term describes the proper birth-jump process. The function $\beta(u)$ is a proliferation rate at location y, and S is the redistribution kernel for newly generated individuals at y to jump to x.

Or even systems

$$\begin{cases} u_t - D_1\Delta u = u\left(\lambda - \displaystyle\int_\Omega K_{11}(x,y)u(y,t)dy - \int_\Omega K_{12}(x,y)v(y,t)dy\right) \\ v_t - D_2\Delta v = v\left(\mu - \displaystyle\int_\Omega K_{21}(x,y)u(y,t)dy - \int_\Omega K_{22}(x,y)v(y,t)dy\right) \end{cases}$$

Hillen, Enderling and Hahnfeldt (2013) proposed the following model to explain "tumor growth paradox":

$$\left\{\begin{array}{rcl} u_t & = & D_1\Delta u + \delta\gamma \displaystyle\int_\Omega K(x,y,p(x,t))u(y,t)dy \\ v_t & = & D_2\Delta v - \alpha v + \rho \displaystyle\int_\Omega K(x,y,p(x,t))v(y,t)dy \\ & & +(1-\delta)\gamma \displaystyle\int_\Omega K(x,y,p(x,t))u(y,t)dy, \end{array}\right.$$

where u, v denote cancer stem cells (CSCs) and nonstem tumor cells (TCs), and $p = u + v$.

In tumors growth models, it appears the equation

$$u_t = d_1 \Delta u - \underbrace{\nabla \cdot (u \nabla v)}_{chemotaxis} + f(u, v)$$

where u and v denote (CE), and TAF.

Consider

$$u_t = d_1 \Delta u - \nabla \cdot (u K \star v) + f(u, v)$$

where

$$K \star v = \int_\Omega K(x, y) v(y) dy.$$

Non-local models: pure nonlocal diffusion

We don't consider in this talk equations with pure non-local equation

$$u_t = \int_\Omega S(x, y, u(x,t))\beta(u(y,t))u(y,t)dy - \delta(u(x,t))u(x,t)$$

Hutson, Martínez, Mischaikow and Vickers (2003), Coville (2004..), García-Melián, Rossi (2008...)

Population dynamics and non-local terms

Our main problem is to analyze

$$\begin{cases} u_t - \Delta u = u\left(\lambda - \int_\Omega K(x,y)u^p(y)dy\right) & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0 & \text{in } \Omega, \end{cases} \tag{2}$$

and the corresponding stationary problem

$$\begin{cases} -\Delta u = u\left(\lambda - \int_\Omega K(x,y)u^p(y)dy\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $p > 0$, $\lambda \in \mathbb{R}$ and K is a continuous function.

$$
\begin{cases}
u_t - \Delta u = u\left(\lambda - \int_\Omega K(x,y)u^p(y)dy\right) & \text{in } \Omega \times (0,\infty),\\
u = 0 & \text{on } \partial\Omega \times (0,\infty),\\
u(x,0) = u_0 & \text{in } \Omega.
\end{cases}
\tag{4}
$$

We can prove:

- If $K \geq 0$, there exists a unique positive solution of (4) for all $t > 0$.
- If $K < 0$, the unique positive solution of (4) blows up in finite time.

Stationary non-local problem

Consider

$$\begin{cases} -\Delta u = u\left(\lambda - \int_\Omega K(x,y)u^p(y)dy\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

with $K(x,y) > 0$ or $K(x,y) < 0$.

Main difficulties:

1. (5) has not a variational structure and so we can not apply the powerful tool of "variational methods" to attack (5).
2. The linearized operator of (5) at a stationary solution is an integral-differential operator and it will not be self-adjoint.

The lack of these properties does not necessarily imply a change on the behaviour of the equation, but it makes the study harder.

Variational structure

In the local case

$$-\Delta u = f(x, u)$$

we have a variational problem associated

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega F(x, v), \quad F(x, s) = \int_0^s f(x, s) ds.$$

In the non-local case: this is not true in general.

We can apply it for particular case, for example

$$-\Delta u = f(u)\left[\int_\Omega F(u)\right]^p$$

with

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \frac{1}{p+1}\left[\int_\Omega F(v)\right]^{p+1},$$

Linearization:

Let u_0 a positive solution of the local problem

$$-\Delta u_0 = f(x, u_0),$$

then the linearization around u_0 is:

$$-\Delta v - f_u(x, u_0)v = \lambda v.$$

Then, there exists a sequence of eigenvalues $\{\lambda_i\}_i \subset \mathbb{R}$ and the principal eigenvalue λ_1 is simple and it is the unique eigenvalue with positive eigenfunction associated φ_1.

Linearization:

Let u_0 a positive solution of the non-local problem

$$-\Delta u_0 = f(x, u_0, B(u_0)), \quad B(u_0) = \int_\Omega u_0^p,$$

then the linearization around u_0 is:

$$-\Delta v - f_u(x, u_0, B(u_0))v - f_v(x, u_0, B(u_0))p \int_\Omega u_0^{p-1} v = \lambda v.$$

What can we say about this eigenvalue problem (non self-adjoint)?

The eigenvalue problem:

$$(EP) \quad \begin{cases} -\Delta v + m(x)v + \int_\Omega K(x,y)v(y)dy = \lambda v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem

Assume that $m \in L^\infty(\Omega)$ and $K \in L^\infty(\Omega \times \Omega)$, $K \leq 0$, $K \neq 0$. Then, there exists a principal eigenvalue of (EP),

$$\lambda_1 = \lambda_1(m, K),$$

which is real, simple, it has an associated positive eigenfunction and it is the unique eigenvalue of (EP) having an associated eigenfunction without change of sign. Moreover, any other eigenvalue λ of (EP) satisfies $\lambda_1 < Re(\lambda)$.

Principal eigenvalue and maximum principle

It holds

$$\boxed{\lambda_1 > 0 \iff \text{ Maximum principle,}}$$

that is, given $f > 0$ and the solution v of

$$-\Delta v + m(x)v + \int_\Omega K(x,y)v(y)dy = f(x),$$

then $v > 0 \iff \lambda_1 > 0$.

Principal eigenvalue, maximum principle and open problems

1. What happens if $K > 0$? OPEN PROBLEM!!!!
2. Given $f \geq 0$, $f \neq 0$ there exists $K > 0$ such that the solution v of the linear problem becomes negative in some part of Ω. This implies:
   - Maximum principle does not hold in general.
   - The sub-supersolution method can not applied. OPEN PROBLEM!!!!
3. What happens if $\|K\|_\infty$ large? OPEN PROBLEM!!!!

The logistic equation with non-local term

$$(P) \qquad \begin{cases} -\Delta u = u\left(\lambda - \displaystyle\int_\Omega K(x,y)u^p(y)dy\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $K(x,y) \geq 0$, $K \neq 0$, $\lambda \in \mathbb{R}$ and $p > 0$.

The local logistic equation

$$\begin{cases} -\Delta u = u(\lambda - a(x)u^p) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\lambda \in \mathbb{R}$, $p > 0$, $\underline{a(x) \geq 0}$ and $\Omega_0 = int\{x \in \Omega : a(x) = 0\} \neq \emptyset$.

Then:

- $u \equiv 0$ is solution for all $\lambda \in \mathbb{R}$.
- There exists a positive solution if and only if $\lambda_1 < \lambda < \lambda_1^{\Omega_0}$. Moreover, the positive solution is unique, denoted by $u^* > 0$.

Furthermore, it is globally stable, that is,

1. If $\lambda < \lambda_1$ we have that $u(x,t) \to 0$ as $t \to \infty$,
2. If $\lambda_1 < \lambda < \lambda_1^{\Omega_0}$ we have that $u(x,t) \to u^*(x)$ as $t \to \infty$.
3. If $\lambda > \lambda_1^{\Omega_0}$ we have that $u(x,t) \to \infty$ as $t \to \infty$.

A remark

Consequence (with respect to the spatial dependence):

- fixed a growth rate of the species, the species coexist if the domain Ω is large, and goes to the extinction if Ω is small.

This confirms ecological results:

Larger islands should be easier to find and colonize, and they should support larger populations which are less susceptible to extinction.

Problem: calculate λ_1.

The logistic equation with non-local term

We come back to our problem: $K(x,y) \geq 0$, consider

$$(P) \qquad \begin{cases} -\Delta u = u \left(\lambda - \int_\Omega K(x,y) u^p(y) dy \right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda \in \mathbb{R}$ and $p > 0$.

1. $(\lambda_1, 0)$ is a bifurcation point from the trivial solution.
2. There exists an unbounded continuum $\mathcal{C}$ of positive solutions emanating from $(\lambda_1, 0)$.
3. There does not exist positive solution for $\lambda \leq \lambda_1$.
4. For any $\Lambda > 0$, there exists $r > 0$ such that: if $(\lambda, u) \in \mathcal{C}$ and $\lambda \leq \Lambda$, we should have $\|u\| \leq r$??

We introduce the class $\mathcal{K}$, which is formed by functions $K : \Omega \times \Omega \to \mathbb{R}$ verifying:

i) $K \in L^\infty(\Omega \times \Omega)$ *and* $K(x, y) \geq 0$ *for all* $x, y \in \Omega$

ii) *If* w *is measurable and*

$$\int_{\Omega\times\Omega} K(x, y)|w(y)|^p w(x)^2 dxdy = 0,$$

then $w = 0$ *a.e in* Ω.

Theorem

Suppose that $K \in \mathcal{K}$. *Then problem* (P) *has a positive solution if, and only if,* $\lambda > \lambda_1$.

(C. Alves, M. Delgado, M. A. Souto, A. Suárez, 2015)

Remark

1. If kernel K does not belong to $\mathcal{K}$, then there exists a measurable $A \subset \Omega$ such that $K = 0$ a.e. in $A \times A$.
2. Suppose that $K(x, y) = 0$ in U for any $y \in \Omega$. Suppose that ∂U is C^1. For any $\lambda_1 < \lambda < \lambda_1(U)$, there exists a positive solution u for the problem (P). Moreover, (P) does not have any positive solution for $\lambda \geq \lambda_1(U)$.
3. In any case, OPEN PROBLEM: UNIQUENESS AND STABILITY?????

Comments:

1. *The non-local logistic equation with non-linear diffusion:*

$$\begin{cases} -\Delta u^m = u \left(\lambda - a(x) \int_\Omega u^p \right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with

$$1 < m, \qquad p > 0,$$

works with F. J. Correa and M. Delgado.

2. *Non-local boundary conditions:*

$$\begin{cases} -\Delta u = \lambda u - u^p & \text{in } \Omega, \\ \mathcal{B}u = \int_\Omega K(x)u(x)dx & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $\lambda \in \mathbb{R}$, $p > 1$, $K \in C(\overline{\Omega})$, $K \geq 0$, $K \neq 0$.

Comments:

3. In the local case,

$$-\Delta u = \lambda u^\beta + u^p$$

it is well known that we have a priori bounds (and so existence of positive solution for $\lambda < \lambda_1$) if for instance $\beta = 1$ and

$$1 < p < \frac{N+2}{N-2};$$

and no existence of solution for $p \geq (N+2)/(N-2)$.
However, in the non-local case

$$-\Delta u = \lambda u^\beta + \int_\Omega u^p$$

we have obtained a priori-bounds (Correa & Suárez (2012)) if:

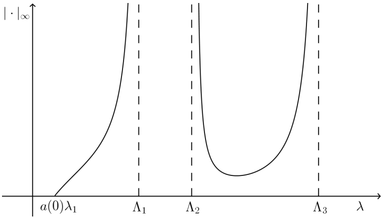
- $\beta = 1$ and for all $p > 1$; or,
- $1 < \beta < \frac{N+2}{N-2}$ and for all $p > 0$; or
- $\beta \geq \frac{N+2}{N-2}$ and $p > (N/2)(\beta - 1)$.

4. Non-local diffusion:

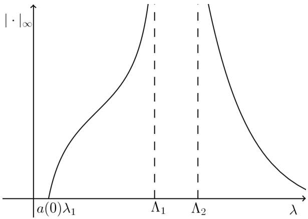
$$-a\left(\int_\Omega q(x)u\right)\Delta u = u(\lambda - b(x)u),$$

where a is a positive and regular function, see Chipot (1996), Chipot & Lovat (1997), Chipot &Correa (2009), etc.....

Comments:



$|\cdot|_\infty$
$a(0)\lambda_1$
Λ_1
Λ_2
Λ_3
λ



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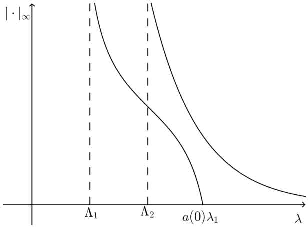

$|\cdot|_\infty$
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