

ESCUELA DE DOCTORADO DE LA UCA  
V Jornadas Doctorales  
Programa de Doctorado en Matemáticas  
**Post-quantum Cryptography**

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UPC/BSC

19/11/2019

## Acknowledgements

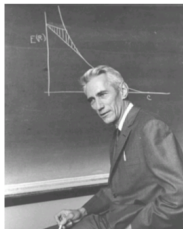
To **Jaume Puigbò**, for his wide aperture view on scientific and technological developments and his unfailing generosity in sharing his findings.

To **Narcís Sayols**, for his collaboration in computational endeavors and particularly in the development of the P<sub>Y</sub>ECC system.

## The digital era

*A Mathematical Theory of Communication* (1948)

- Mathematical foundations of communication systems.
- Definition of information (*bit*), of channel capacity and of error-correcting codes.
- Source and channel coding theorems.
- **Founder of modern mathematical cryptography**



**Claude Shannon** (1916-2001)

- ❖ Master's thesis (1937): logic circuits
- ❖ II WWII: Cryptography



Euclides



Euler



Leibniz



Gauss



Galois



Ada Lovelace



Church



von Neumann



Turing



Shannon



Diffie-Hellman-Merkel



McEliece



Cocks



Rivest-Shamir-Adleman



Euclid (325 BC – 265 BC).

Pierre de Fermat (1607-1665). Blaise Pascal (1623-1662). Gottfried Leibniz (1646-1716).

Leonhard Euler (1707-1783). Karl Friedrich Gauss (1777-1855). Charles Babbage (1791-1871).

Évariste Galois (1811-1832). Ada Lovelace (1815-1852). George Boole (1815-1864). Alonzo Church (1903-1995).

John von Neuman (1903-1957). Alan Turing (1912-1954). Claude Shannon (1916-2001). Elwyn Berlekamp (1940-2019).

Whitfield Diffie (1944). Ralph Merkel (1952). Martin Hellman (1945). Robert McEliece (1942). Ronald Rivest (1947). Adi Shamir (1952). Leonard Adleman (1945). Clifford Cocks (1950).

- **Remarks on mathematical computation.**
- **RSA in a nutshell.**
- **The code-based McEliece cryptosystem.**
- **Quantum computing.**
- **Post-quantum cryptography.**
- **Urmila Mahadeva, for outstanding achievements.**
- **Further references.**

# Remarks on mathematical computation

The PyECC system · How can you get it · Examples

- A purely Python computational environment. Small, powerful, and free. Initially designed to cover the computational needs of a book such as [1] (*Error-Correcting Codes. A computational Primer*).
- It is being extended to meet other related purposes, such as, among others, the second edition of [1] (planned to have additional chapters on *post-quantum cryptography*, *quantum codes*, and *convolutional codes*) and [2] (A second edition of *Using intersection theory* and planned to cover a wide range of computations in intersection theory and enumerative geometry).



- For more information, including instructions for how to install it, visit the Web page [PyECC](#).

## Py Error-Correcting Coding (PyECC)

S. Xambó

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### Description

This is a venture in collaboration with [Narcís Sayols Baixeras](#). Its aim is to produce a [Python](#) package ([PyECC](#)) enabling the construction, coding and decoding of error-correcting codes and make it freely available for students, teachers and researchers.

Initially (October 2015) the idea was to match the functionality of the CC system developed to deal with the computational tasks related to the book [Xambó-2003](#), but we soon found that we could go beyond that system in several directions. The purpose of this page ([work in progress](#)) is to document the current state of the project.

### INDEX

- [FUNCTION SUMMARY](#)
- [PYECC JUPYTER NOTEBOOKS](#)
- [INSTALLING INSTRUCTIONS](#)
- [STRUCTURE](#)
- [VERSION HISTORY](#)

- For online resources related to this talk, visit the Web page [PyACA](#), which provides access to Python sources and Jupyter notebooks that are companion materials to the paper [3].

## Index of links

- *Modular arithmetic* [[py](#) | [nb](#)]
- *Computation of  $I_q(m)$*  [[py](#) | [nb](#)]
- *Construction of rings and fields* [[py](#) | [nb](#)]
- *Vectors and matrices* [[py](#) | [nb](#)]
- *Random permutation matrix* [[py](#) | [nb](#)]
- *$rd\_GL(k,F)$*  [[py](#) | [nb](#)]
- *The Hamming code [7,4,3]* [[py](#) | [nb](#)]
- *Coding and decoding the Hamming code [7,4,3]* [[py](#) | [nb](#)]
- *Example of blow and prune* [[py](#) | [nb](#)]
- *Example of a Goppa code* [[py](#) | [nb](#)]
- *An illustration of the McEliece system, step by step* [[py](#) | [nb](#)]

```
def NOT(j, x):  
    x[j] -= 1  
    return x
```

# Controlled-not. For  $j=i$ ,  $x[i]$  is set to 0.

```
def CNOT(i,j, x):  
    x[j] += x[i]  
    return x
```

# Remark: for  $i=j$  it agrees with  $\text{CNOT}(j,k, x)$

```
def TOFFOLI(i,j,k, x):  
    x[k] += x[i]*x[j]  
    return x
```

**Fact:** Any classical computation can be embedded in a reversible computation (see [4]) and any reversible computation can be achieved with a sequence of Toffoli gates.

```
# To pick a random n-digit prime number
def rd_prime(n):
    while True:
        r = ri(n) # a random integer of n digits
        if even(r): r += 1
        if is_prime(r): return r

rd_prime(10) => 31716 43721
rd_prime(100) =>
75549 06889 06549 88619 83249 61562 84269 54436 19389 71081
98583 48723 01531 63041 11630 94137 11913 03811 57707 35373
```

The density of  $n$ -digit prime numbers is  $\sim 0.48/(n+1)$ , which roughly means an average of  $2n$  trials to produce a prime number.

```
# x, the message, is a binary vector
# p is a random binary vector of the same length as x.
# The encrypted message is  $y = x+p$ 
# Decryption is  $y + p = x + p + p = x$ 
```

```
def one_time_pad(x):
    n = len(x)
    p = rd_vector(Zn(2),n)
    return (x+p,p)
```

```
x = rd_vector(Zn(2),25)
one_time_pad(x) =>
([0,0,0,0,0,1,0,0,1,0,1,1,0,0,1,0,0,1,0,0,0,1,1,1,0],
 [1,1,1,0,1,0,1,1,1,1,1,1,0,1,1,0,1,1,1,1,0,0,0,0,0])
```

```
# Pseudo-random generator of n bits
# (Blum-Blum-Shup, 1996)
def PRG(n,d=30):
    N = rd_prime(d)*rd_prime(d)
    x = rd_int(1,N-1)
    s = str(x%2)
    for _ in range(1,n):
        x = x**2 % N
        s += str(x%2)
    return s
```

```
PRG(40) => '1000101110101010111000111011111010011101'
```

The function `permutation_matrix(n)` creates a random permutation matrix of order  $n$ .

```
def rd_permutation_matrix(n):  
    N = list(range(n))  
    p = rd_permutation(n)  
    P = matrix(ZZ(), n, n)  
    for j in range(n):  
        P[j, p[j]] = 1  
    return P
```

`rd_permutation_matrix(5) =>`

```
[[1 0 0 0 0]  
 [0 0 0 0 1]  
 [0 1 0 0 0]  
 [0 0 0 1 0]  
 [0 0 1 0 0]]
```

The construction of uniformly random invertible matrix  $S$  of order  $k$  with entries in a finite field  $F$  is a bit involved (see [3], particularly the function `rd_GL(k,F)`). Instead, let us consider an easier function `scramble_matrix(k,A)` which creates a random  $S \in A(k)$  with  $\det(S) = \pm 1$ . Note: `rd(A)` returns an element of  $A$  selected uniformly at random.

```
def scramble_matrix(k,A):
    U = matrix(A,k,k)
    L = matrix(A,k,k)
    for i in range(k):
        U[i,i] = L[i,i] = 1
        for j in range(i+1,k):
            U[i,j] = rd(A)
            L[j,i] = rd(A)
    P = permutation_matrix(k)
    return P*L*U
```



# RSA in a nutshell

RSA pairs of prime numbers · RSA keys ·  
Encryption and decryption · RSA numbers

```
def rsa_pair(n,m=3):
    while True:
        p = rd_prime(n); q = rd_prime(n)
        if igcd(m,p-1)>1 or igcd(m,q-1)>1 or p==q:
            continue
        else: break
    return (p,q)
```

```
rsa_pair(20) =>
(21569513085908660339, 15526103590621876157)
```

```
rsa_pair(20,17) =>
(33996414417889718849, 56610744900409885459)
```

Fix a positive integer  $n$  (in practice today, a three digit number, say 120).

$(p, q)$  an RSA pair of  $n$ -digit prime numbers for some  $m$  (say  $m = 3$  for concreteness).

$k$  the inverse of  $m \bmod (p - 1)(q - 1)$

**Secret keys:**  $(p, q, m, k)$ .

**Public keys:**  $(N, m)$ , where  $N = pq$ .

$(p, q) = \text{rsa\_pair}(10) \Rightarrow (2952363761, 6768633203)$

$k = \text{inverse}(3, (p-1)*(q-1)) \Rightarrow 13322311580211706347$

$N = p*q = 19983467380038556483$

$x$ : a number in  $(1, N - 1)$ , the *message*.

**Encryption:**  $y = x^m \pmod N$ .

**Decryption:**  $z = y^k \pmod N$ .

**Fact:**  $z = x$ .

*Proof:*  $y^k = (x^m)^k = x^{mk} = x^{1+r(p-1)(q-1)} = x$ , as  $(p-1)(q-1) = \varphi(N)$  and hence  $x^{(p-1)(q-1)} = 1$  (Euler).

## Security

- *RSA is secure if factoring  $N$  is unfeasible.*
- The best known algorithms factor  $N$  (a  $2n$ -digit number) in subexponential expected time  $e^{(1+o(1))r(n)}$ ,  $r(n) = \sqrt{2n \log(2n)}$ .

$n$	100	200	300	400	500	1000
$\lfloor \log_2 e^{r(n)} \rfloor$	46	70	89	105	119	177

RSA-100	RSA-170 <sup>174</sup>	RSA-230	RSA-290 *	RSA-360 *	RSA-450 *
RSA-110	<u>RSA-576</u>	RSA-232 *	RSA-300 *	RSA-370 *	RSA-460 *
RSA-120	RSA-180	<u>RSA-768</u> <sup>232</sup>	RSA-309 *	RSA-380 *	<u>RSA-1536</u> <sup>463</sup>
<u>RSA-129</u>	RSA-190 <sup>193</sup>	RSA-240 *	<u>RSA-1024</u> <sup>309</sup>	RSA-390 *	RSA-470 *
RSA-130	<u>RSA-640</u>	RSA-250 *	RSA-310 *	RSA-400 *	RSA-480 *
RSA-140	RSA-200	RSA-260 *	RSA-320 *	RSA-410 *	RSA-490 *
RSA-150	RSA-210 <sup>212</sup>	RSA-270 *	RSA-330 *	RSA-420 *	RSA-500 *
RSA-155	<u>RSA-704</u>	<u>RSA-896</u> <sup>270</sup>	RSA-340 *	RSA-430 *	<u>RSA-617</u>
RSA-160	RSA-220	RSA-280 *	RSA-350 *	RSA-440 *	<u>RSA-2048</u> <sup>617</sup>

**RSA Factoring Challenge:** 1991-2007. Each number is the products of two primes of similar size. Those underlined in red are quoted in bits, with the number of decimal digits written in grey above them. The numbers with \* have not been factored yet.

Source: [https://en.wikipedia.org/wiki/RSA\\_numbers](https://en.wikipedia.org/wiki/RSA_numbers).

## RSA-768 <sup>232</sup> [edit]

RSA-768 has 232 decimal digits (768 bits), and was factored on December 12, 2009 over the span of two years, by Thorsten Kleinjung, Kazumaro Aoki, Jens Franke, [Arjen K. Lenstra](#), Emmanuel Thomé, Pierrick Gaudry, Alexander Kruppa, [Peter Montgomery](#), Joppe W. Bos, Dag Arne Osvik, Herman te Riele, Andrey Timofeev, and [Paul Zimmermann](#).<sup>[36]</sup>

```
RSA-768 = 12301866845301177551304949583849627207728535695953347921973224521517264005
07263657518745202199786469389956474942774063845925192557326303453731548268
50791702612214291346167042921431160222124047927473779408066535141959745985
6902143413
```

```
RSA-768 = 33478071698956898786044169848212690817704794983713768568912431388982883793
878002287614711652531743087737814467999489
× 36746043666799590428244633799627952632279158164343087642676032283815739666
511279233373417143396810270092798736308917
```

The CPU time spent on finding these factors by a collection of parallel computers amounted approximately to the equivalent of almost 2000 years of computing on a single-core 2.2 GHz AMD Opteron-based computer.<sup>[36]</sup>

## RSA-240 [\[edit\]](#)

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RSA-240 has 240 decimal digits (795 bits), and has not been factored so far.

```
RSA-240 =  
12462036678171878406583504460810659043482037465167880575481878888328966680118821  
08550360395702725087475098647684384586210548655379702539305718912176843182863628  
46948405301614416430468066875699415246993185704183030512549594371372159029236099
```

## RSA-250 [\[edit\]](#)

---

RSA-250 has 250 decimal digits (829 bits), and has not been factored so far.

```
RSA-250 = 2140324650240744961264423072839333563008614715144755017797754920881418023447  
1401366433455190958046796109928518724709145876873962619215573630474547705208  
0511905649310668769159001975940569345745223058932597669747168173806936489469  
9871578494975937497937
```

# McEliece cryptosystems

Ingredients of a McECS · Encryption · Decryption ·  
Construction of McECS · Security analysis



- $F = F_q$ , a finite field of cardinal  $q$  (*base field*). The most important case will be  $F = \mathbf{Z}_2$ .
- $k$  a positive integer. The vectors of  $F^k$  are called *information vectors*, or *messages*.
- $n > k$  an integer. The vectors of  $F^n$  are called *transmission vectors*.

## Notations

If  $\mathbf{x} \in F^n$ , we let  $|\mathbf{x}|$  denote the number of non-zero components of  $\mathbf{x}$  and we say that it is the *weight* of  $\mathbf{x}$ .

$F(r, s)$  denotes the space of matrices of type  $r \times s$  with entries in  $F$  and  $F(r) = F(r, r)$ .

A receiving user needs the following data:

- $G \in F(k, n)$  such that  $\text{rank}(G) = k$ ;
- $S \in F(k)$  invertible and chosen uniformly at random;
- $P \in F(n)$  a random permutation matrix;
- $t$ , a positive integer; and
- $g : X \rightarrow F^k$ ,  $X \subseteq F^n$ , such that for any  $\mathbf{u} \in F^k$  and all  $\mathbf{e} \in F^n$  with  $|\mathbf{e}| \leq t$ ,

$$\mathbf{x} = \mathbf{u}G + \mathbf{e} \in X \quad \text{and} \quad g(\mathbf{x}) = \mathbf{u}. \quad (1)$$

The map  $g$  is called an  *$t$ -error-correcting  $G$ -decoder*, or simply *decoder*, and the vectors of  $X$  are said to be  *$g$ -decodable*.

- **Private key:**  $\{G, S, P\}$ .
- **Public key:**  $\{G', t\}$ , where  $G' = SGP$ .

## Encryption protocol

The protocol that a user has to follow to encrypt and send a message  $u$  to the user whose public key is  $\{G', t\}$  consists of two steps:

- Random generation of a transmission vector  $e$  of weight  $t$ ;
- Sending the vector  $x = uG' + e = uSGP + e$  to that user.

## Decryption protocol

Consists of four steps that only use private data of the receiver and the vector  $\mathbf{x}$  sent by the emitter:

- Set  $\mathbf{y} = \mathbf{x}P^{-1}$ , so that  $\mathbf{y} = (\mathbf{u}S)G + \mathbf{e}P^{-1}$ .
- Set  $\mathbf{x}' = g(\mathbf{y})$ . Since  $P$  is a permutation matrix,  $|\mathbf{e}P^{-1}| = |\mathbf{e}| = t$ , and hence  $\mathbf{x}'$  is well defined, as  $g$  corrects  $t$  errors. The result is  $\mathbf{x}' = (\mathbf{u}S)G$ , which says that  $\mathbf{x}'$  is the linear combination of the rows of  $G$  with coefficients  $\mathbf{u}' = \mathbf{u}S$ .
- Since  $G$  has rank  $k$ ,  $\mathbf{u}'$  is uniquely determined by  $\mathbf{x}'$  and can be obtained by solving the system of linear equations  $\mathbf{x}' = \mathbf{u}'G$ , where  $\mathbf{u}'$  is the unknown vector.
- Let  $\mathbf{u} = \mathbf{u}'S^{-1}$ , which agrees with the message sent by the emitter.

- $F = F_2$  (most constructions also for  $F_q$ ,  $q > 2$ ).
- $\bar{F} = F_{q^m}$ ,  $m$  a positive integer. If  $\beta \in \bar{F}$ ,  $[\beta]$  will denote the column vector of its components with respect to a basis of  $\bar{F}/F$ .
- $\alpha = \alpha_1, \dots, \alpha_n \in \bar{F}$  distinct elements, so that  $n \leq q^m$ .
- $p \in \bar{F}[X]$  a polynomial of degree  $r > 0$  such that  $p(\alpha_j) \neq 0$  ( $j = 1, \dots, n$ ).
- Set  $h_j = 1/p(\alpha_j)$  ( $j = 1, \dots, n$ ) and

$$\bar{H} = \begin{pmatrix} h_1 & \cdots & h_n \\ h_1\alpha_1 & \cdots & h_n\alpha_n \\ \vdots & & \vdots \\ h_1\alpha_1^{r-1} & \cdots & h_n\alpha_n^{r-1} \end{pmatrix} \in \bar{F}(r, n).$$

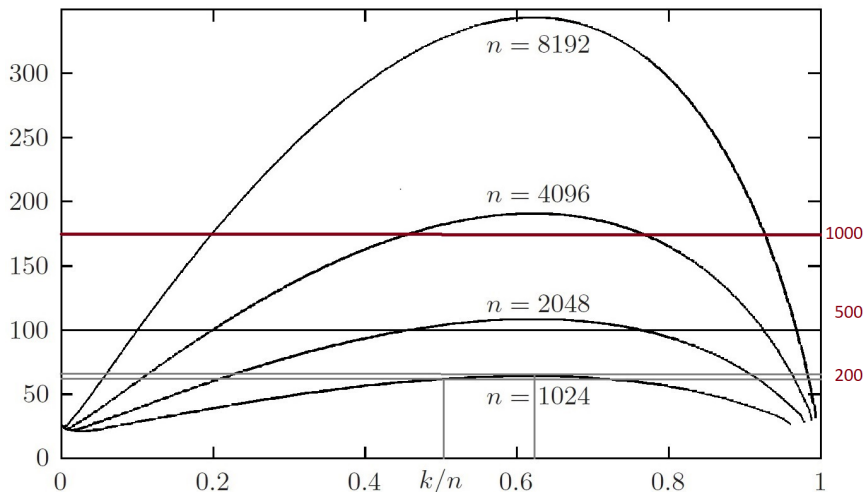
- Let  $H \in F(r', n)$  be the result of replacing each entry  $\beta$  of  $\bar{H}$  by  $[\beta]$  (this yields a matrix  $[H] \in F(mr, n)$ ), followed by deleting from  $[H]$  any row that is in the span of the previous ones. Note that  $r' \leq mr$ . It also holds that  $r \leq r'$ , as the  $\langle H \rangle_{\bar{F}} = \langle \bar{H} \rangle_{\bar{F}}$ .
- Let  $\Gamma = \Gamma(p, \alpha) = \{\mathbf{x} \in F^n : \mathbf{x}H^T = 0\}$ . It is a **code** of type  $[n, k = n - r']$ . This code is called the *classical Goppa code* associated to  $p$  and  $\alpha$ .
- We have  $n - mr \leq k \leq n - r$ .
- **Fact:** If  $G \in F(k, n)$  is a generating matrix of  $\Gamma$ , *there is  $G$ -decoder that corrects  $r$  errors ( $r/2$  for  $q > 2$ ) provided  $p$  has no multiple roots in  $\bar{F}$* . See, for example, [1, P.4.7]

## Practical specification

- Let  $\alpha$  be the set of elements of  $\bar{F}$ . Hence  $n = 2^m$ .
- Let  $p \in \bar{F}[X]$  be a monic irreducible of degree  $t > 1$ . Then  $p$  has no roots in  $\bar{F}$  and so a generating matrix  $G$  of  $\Gamma(p, \alpha)$  *has a decoder  $g$  that corrects  $t$  errors*.

This ends the theoretical construction of a McECS with the following parameters:

- $n = 2^m$ , where  $m$  is any positive integer, and  $p$  is monic irreducible of degree  $t$ .
- $\bar{H} \in \bar{F}(t, n)$  and  $G \in F(k, n)$ , where  $k = n - \text{rank}(H)$  ( $n - tm \leq k \leq n - t$ ).
- Original example:**  $m = 10$ ,  $n = 1024$ ,  $t = 50$ ,  $k = 524$  (in this case  $k = n - tm$ , the minimum possible given  $m$  and  $t$ ).



Horizontal axis  $R = k/n$ , WF curves for  $n = 2^j$ ,  $j = 10, \dots, 13$ . The red line represents the WF needed to break RSA with 1000-digit prime numbers. The similar 500-digit and 200-digit levels are also shown. The latter is comparable to the original McECS.





Planck



Einstein



Schrödinger



de Broglie



Dirac



Pauli



Feynman



Shor

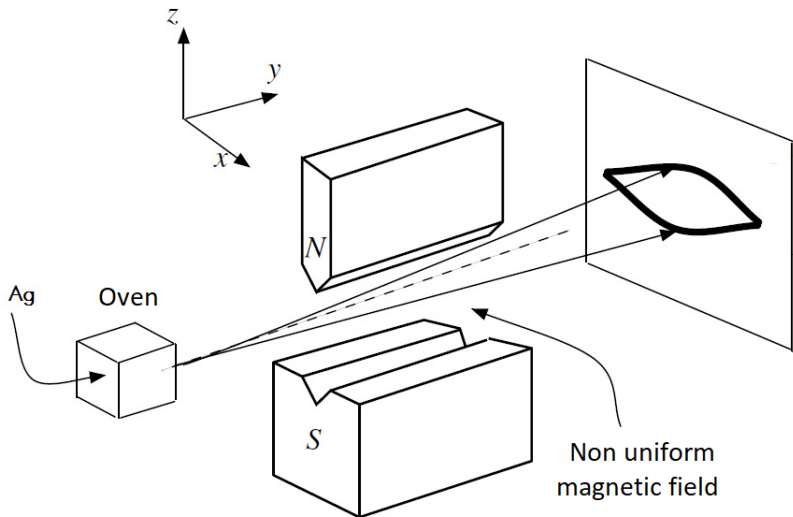
Max Planck (1858-1947), Albert Einstein (1879-1955), Erving Schrödinger (1887-1966), Louis de Broglie (1892-1987).

Paul Dirac (1902-1984), Wolfgang Pauli (1909-1958).

Richard Feynman (1918-1988), Peter Shor (1959).

# Quantum computing

$q$ -bits ·  $q$ -registers ·  $q$ -computations ·  $q$ -gates ·  
 $q$ -algorithms · Shor's factoring  $q$ -algorithm ·  
Grover's searching  $q$ -algorithm · The power of  
 $q$ -computing.



**Stern-Gerlach** experiment, which uncovered the quantum nature of the electron *spin*.

- To move from classical to quantum computation, replace  $B = \{0, 1\}$ , the set of classical bits, by all their ‘superpositions’ (linear combinations with complex coefficients), i.e. by the complex space  $E = E^{(1)}$  generated by  $B$ .

(Superpositions of waves occur in classical physics, particularly in many wave phenomena, and in the related notion of polarization states of electromagnetic waves. Its general validity in the context of quantum physics is one of the main tenets of this theory)

Thus the elements of  $E$  have the form  $\psi = \psi_0 \mathbf{e}_0 + \psi_1 \mathbf{e}_1$ , where  $\mathbf{e}_0$  and  $\mathbf{e}_1$  is the basis corresponding to 0 and 1 and  $\psi_0, \psi_1 \in \mathbf{C}$ .

The vectors  $\psi$  of norm 1 form a sphere  $S^3 \subset E^{(1)} \simeq \mathbf{R}^4$  and are called *Pauli spinors* ( $|\psi|^2 = |\psi_0|^2 + |\psi_1|^2$ ).

- The  $q$ -bit *state* corresponding to a spinor  $\psi$  is the point (in *Dirac's ket notation*)  $|\psi\rangle \in S^2 \subset \mathbf{R}^3$  defined by the formulas

$$x = (\bar{\psi}_0\psi_1 + \psi_0\bar{\psi}_1), y = -i(\bar{\psi}_0\psi_1 - \psi_0\bar{\psi}_1), z = (\bar{\psi}_0\psi_0 - \bar{\psi}_1\psi_1).$$

The map  $S^3 \rightarrow S^2$ ,  $\psi \mapsto |\psi\rangle$  is the *Hopf fibration*.

In fact,  $|\psi\rangle = |\psi'\rangle$  if and only if  $\psi' = \xi\psi$  for some unit complex number  $\xi$ , a relation that we will denote  $\psi \sim \psi'$ .

■ If we set

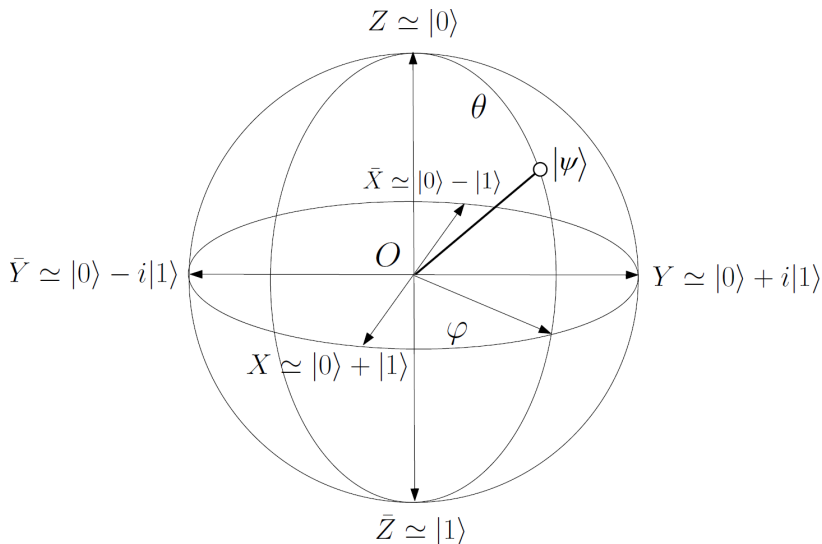
$$\psi_{\phi,\theta} = e^{-i\phi/2} \cos \frac{\theta}{2} \mathbf{e}_0 + e^{i\phi/2} \sin \frac{\theta}{2} \mathbf{e}_1$$

( $0 \leq \phi < 2\pi$ ,  $0 \leq \theta \leq \pi$ ), then  $S_{\phi,\theta} = |\psi_{\phi,\theta}\rangle \in S^2$  is the point at longitude  $\phi$  and colatitude  $\theta$ . In particular,

$$\pm X = (\pm 1, 0, 0) = \left| \frac{\sqrt{2}}{2} (\mathbf{e}_0 \pm \mathbf{e}_1) \right\rangle,$$

$$\pm Y = (0, \pm 1, 0) = \left| \frac{\sqrt{2}}{2} (\mathbf{e}_0 \pm i\mathbf{e}_1) \right\rangle,$$

$Z = |\mathbf{e}_0\rangle = (0, 0, 1)$  (North pole),  $-Z = |\mathbf{e}_1\rangle = (0, 0, -1)$  (South pole).



$$\psi = e^{-i\varphi/2} \cos\left(\frac{\theta}{2}\right) |0\rangle + e^{i\varphi/2} \sin\left(\frac{\theta}{2}\right) |1\rangle$$



■ The *measurement* of a  $q$ -bit that is in the state  $S_{\phi,\theta}$ , which corresponds to the *reading* of a classical bit, produces the state  $|e_0\rangle = N = \uparrow$  or the state  $|e_1\rangle = -N = \downarrow$ , and this result is a *random event* with *probabilities*  $p_S(\uparrow) = \cos^2 \frac{\theta}{2}$  and  $p_S(\downarrow) = \sin^2 \frac{\theta}{2}$ .

These states  $\uparrow$  and  $\downarrow$  are classical, in that  $p_{\uparrow}(\uparrow) = p_{\downarrow}(\downarrow) = 1$  (in agreement with the Stern-Gerlach findings), and they are the only states having this property. For states  $S$  on the equator ( $\theta = \pi/2$ ),  $\uparrow$  and  $\downarrow$  are *equiprobable*, and they are the only ones having this property.

- More generally, the model of a *q-register of length n*,  $Q_n$ , is based on replacing  $B^n$  (the space of  $n$ -bit strings) by  $E^{(n)} = \langle B^n \rangle_{\mathbb{C}}$ , the complex vector space with basis  $B^n$ , represented as the tensor product (*q-entanglement* of the  $q$ -bits in the  $q$ -register) of  $E^{(1)}$  with itself  $n$  times.

So the vectors of  $E^{(n)}$  have the form  $\psi = \sum_{b \in B^n} \psi_b e_b$ , where  $e_b = e_{b_1} \otimes \cdots \otimes e_{b_n} \equiv e_{b_1} \cdots e_{b_n}$ . A vector  $\psi$  is *normalized* if  $|\psi|^2 = \sum_{b \in B^n} |\psi_b|^2 = 1$ .

- Each normalized vector  $\psi$  defines a state  $|\psi\rangle$  in the *state space*  $\Sigma_n$  of a  $Q_n$ , with the rule that  $|\psi\rangle = |\psi'\rangle$  if and only if  $\psi' = \xi\psi$  for some unit complex number  $\xi$ . Again, we denote this relation by  $\psi \sim \psi'$ . As we have seen,  $\Sigma_1 = S^2$ .

- The *measuring* of  $Q_n$  in the state  $S = |\psi\rangle$  returns one of the basis states  $|e_b\rangle \equiv |b\rangle$  at random with probabilities  $|\psi_b|^2$ .

- The notion of a classical (reversible) computation on  $n$  bits is replaced by a **unitary** matrix  $U$  of order  $2^n$ , which can be viewed as a unitary transformation  $U : E^{(n)} \rightarrow E^{(n)}$ .
- To any classical computation  $f : B^n \rightarrow B^n$  we can associate the  $q$ -computation  $U_f : E^{(n)} \rightarrow E^{(n)}$  defined by  $e_b \mapsto e_{f(b)}$ . It is a permutation matrix.
- In particular we can regard the logical gates CNOT and TOFFOLI as **quantum gates**. Thus, for instance, CNOT(1,4) leaves  $e_b$  fixed if  $b_1 = 0$  and changes it to  $e_{b'}$  if  $b_1 = 1$ , where  $b'_4 = 1 + b_4$  and otherwise  $b'_i = b_i$ .

- For a  $q$ -bit, the *Hadamard gate*,  $H$ , is defined by  $e_0 \mapsto \frac{\sqrt{2}}{2}(e_0 + e_1)$ ,  $e_1 \mapsto \frac{\sqrt{2}}{2}(e_0 - e_1)$ :

$$H = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

This is a *genuine  $q$ -gate*, as the states  $\pm X$  defined by  $H(e_0)$  and  $H(e_1)$  are not classical.

In the case of  $Q_n$ ,  $H$  can be applied to any  $q$ -bit, or to all, in which case we will denote it by  $H^{(n)}$ . For example:

$$H^{(2)}e_{00} = \frac{1}{2}(e_0 + e_1)(e_0 + e_1) = \frac{1}{2}(e_{00} + e_{01} + e_{10} + e_{11}).$$

Thus, on measuring, the four possible results 00, 01, 10, 11 are equiprobable.

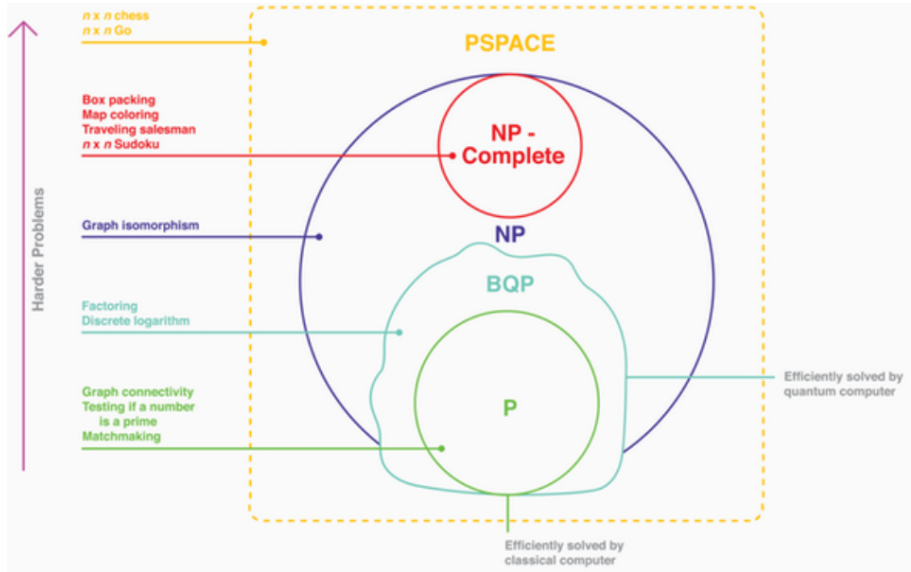
In general,  $H^{(n)}e_{0\dots 0} = \left(\frac{\sqrt{2}}{2}\right)^n \sum_{b \in B^n} e_b$ , a superposition for which all states  $|b\rangle$  are equiprobable. This embodies the so-called *quantum parallelism*.

- A *q-algorithm* is a sequence of Hadamard and Toffoli gates, followed by a measure of the final state, which is the (classical) bit string returned by the algorithm.

*Example.* Initialize  $Q_3$  in the state  $|000\rangle$ . Apply  $H^{(3)}$ . End by a measuring operation. This *q*-algorithm yields a uniform random string of three bits. The generalization to  $Q_n$  is obvious.

- *Shor's q-algorithm factors integers in polynomial time.*
- *Grover's q-algorithm searches an item in a list of size  $N$  in time  $\sqrt{N}$ .*

For a systematic introduction to *q*-computing, including a discussion of these and other algorithms, see [4], and also the references therein.



An excellent discussion about the complexity theory issues is provided by Scott Aaronson's book [7] (*Quantum computing since Democritus*).

# Post-quantum cryptography

Quantum threats to cryptographic protocols.  
McECS as a post-quantum system. Other  
post-quantum protocols. The NIST initiative for  
PQ protocols.

- $q$ -computing poses a fundamental threat to widespread cryptographic systems like RSA.
- In principle, it does not pose a threat to McECS (unless  $P = NP$ ), because the general problem of decoding linear codes is NP complete [8] (Berlekamp, McEliece and van Tilborg, 1978: *On the inherent intractability of certain coding problems*).

However, for the special codes used in McECS it could still exist an astute way of using their structure to crack them. But all the evidence collected in their study so far (see page 32) suggests it is a *post-quantum* protocol, in the sense that no computational power can crack it if the appropriate parameters are used.



In addition to the McECS, there are other systems that may qualify as post-quantum cryptography:

- Hash-based cryptography;
- Code-based cryptography;
- Lattice-based cryptography;
- Multivariate-quadratic-equations cryptography.

See [9], particularly the introductory paper by D. J. Bernstein.

These, and variations on them, are being considered by NIST with the goal set at defining and standardize one or more post-quantum cryptography protocols. See <http://dx.doi.org/10.6028/NIST.IR.8105>.

# Urmila Mahadev

For outstanding achievements



“Urmila Mahadev spent eight years in graduate school solving one of the most basic questions in quantum computation: How do you know whether a quantum computer has done anything quantum at all?” (article by Erika Klarreich, 8 October 2018, in Quantamagazine).

<https://www.quantamagazine.org/graduate-student-solves-quantum-verification-problem-20181008/>

See [10] (U. Mahadev, *Classical Verification of Quantum Computations*).

**Abstract.** We present the first protocol allowing a **classical computer** to interactively **verify the result** of an efficient quantum computation.

We achieve this by constructing a *measurement protocol*, which enables a classical verifier to use a quantum prover as a trusted measurement device.

The protocol forces the prover to behave as follows: (1) the **prover** must construct an  $n$  qubit state of his choice, (2) measure each qubit in the *Hadamard* or standard basis **as directed by the verifier**, and (3) report the measurement results to the verifier.

The soundness of this protocol is enforced *based on the assumption that the learning with errors problem* [one of the post-quantum protocols] is computationally intractable for efficient quantum machines.

# Further references

On mathematical cryptography: [11].

For a study of McECS and its security: [3]

On quantum computing: [12]. For an approach to the  $q$ -bit based on the stereographic projection of  $S^2$ , see [13].

Another important paper of Urmila Mahadev: [14].

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